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MODEL FUNCTIONS WITH NEARLY PRESCRIBED MODULUS

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Abstract. Let Θ be an inner function on the upper half-plane, and let $K_\Theta = H^2 \ominus \Theta H^2$ be the corresponding model subspace. A nonnegative measurable function ω is said to be strongly admissible for K_Θ if there exists a nonzero function $f \in K_\Theta$ with $|f| \asymp \omega$. Certain conditions sufficient for strong admissibility are given in the case where Θ is meromorphic.

§1. Introduction

Let Θ be an inner function on the upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ (recall that a function on $\mathbb{C}^+$ is said to be *inner* if it is analytic and bounded on $\mathbb{C}^+$ and its angular boundary values are of modulus one a.e. on $\mathbb{R}$). The function Θ generates the so-called *model subspace*

$$K_\Theta = H^2 \ominus \Theta H^2$$

of the Hardy space $H^2 = H^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0 \text{ for a.e. } \xi \in (-\infty, 0)\}$ (here $\hat{f}$ is the Fourier transform of f). These spaces play an important part in analysis and its applications to mathematical physics; we refer the reader to the monographs [Cima, N, NF], and to the papers [Bl2, Bl3, BH, BBH, D, HM1, HM2, MNH], which are particularly close to the subject matter of the present article (it is devoted to the moduli of model functions, i.e., of elements of K_Θ). Here, the main attention is paid to the classical case where $\Theta(z) = e^{i\sigma z}$, $\sigma > 0$; then K_Θ turns into $e^{i\sigma z/2} PW_{\sigma/2}$ (by PW_σ we denote the Paley–Wiener space that consists of the entire functions of degree at most σ and square integrable on $\mathbb{R}$).

As experience shows, the natural problem of describing the nonnegative functions ω that coincide a.e. on $\mathbb{R}$ with the modulus of a model function (Θ being fixed) usually admits no satisfactory solution; see [D, HM1]. Even the problem (much simpler at first glance) of finding a nonzero $f \in K_\Theta$ such that

$$|f| \le \omega \quad \text{a.e. on } \mathbb{R} \tag{1.1}$$

for a majorant $\omega : \mathbb{R} \mapsto [0, +\infty)$ fixed in advance, requires fairly deep analytic techniques. For example, the celebrated Beurling–Malliavin multiplier theorem is a result pertaining to the classical case of $\Theta(z) = e^{i\sigma z}, \sigma > 0$.

Definition 1.1. A nonnegative function ω on $\mathbb{R}$ is said to be Θ-admissible (in symbols: $\omega \in \operatorname{Adm}(\Theta)$) if there exists a nonzero function $f \in K_\Theta$ satisfying (1.1).

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A necessary condition for Θ-admissibility is the convergence of the logarithmic integral $\mathcal{L}(\omega)$:

$$\mathcal{L}(\omega) = \int_{\mathbb{R}} \Omega \, d\mathbf{P}, \quad \Omega = -\log \omega,$$

where $d\mathbf{P}$ stands for the Poisson measure on $\mathbb{R}$:

$$d\mathbf{P}(x) = \frac{1}{\pi} \frac{dx}{1+x^2}.$$

We assume that ω is Lebesgue measurable and $0 \le \omega \le 1$ a.e. on $\mathbb{R}$, so that the convergence of the integral $\mathcal{L}(\omega)$ means that $\mathcal{L}(\omega) > -\infty$. In should be noted that this condition never suffices for Θ-admissibility (see [BH]). Sufficient conditions strongly depend on the structure of the generating function Θ; they were studied in the papers [Bl2, Bl3, BH, BBH, HM1, HM2, MNH] already cited.

In the present paper, we consider a subset of $\mathrm{Adm}(\Omega)$ that consists of the so-called *strongly admissible* functions.

Definition 1.2. A function ω is said to be strongly admissible (more precisely, strongly Θ-admissible; in symbols, $\omega \in \mathrm{sAdm}(\Theta)$) if there exists a function $f \in K_\Theta$ and two positive constants C_1, C_2 such that

$$C_1 \omega \le |f| \le C_2 \omega \quad \text{a.e. on } \mathbb{R} \tag{1.2}$$

(for short, we often write $|f| \asymp \omega$ if such C_1, C_2 exist).

We need yet another notation:

$$\mathrm{sADM}(\Theta) = \{-\log \omega, \, \omega \in \mathrm{sAdm}(\Theta)\}.$$

Instead of the nonrealistic problem of finding $f \in K_\Theta$ with prescribed modulus ("$|f| = \omega$ a.e."), the function f in (1.2) solves a weaker problem, specifically, that of finding a function in K_Θ with *nearly* prescribed modulus ($|f|$ is only *comparable* with ω on $\mathbb{R}$: $|f| \asymp \omega$).

The main result of this paper (Theorem 2.4 in §2) describes a fairly wide subset of $\mathrm{sAdm}(\Theta)$ in case Θ is a meromorphic inner function. But we shall start the discussion with a partial and simpler case of that theorem pertaining to the classical situation of $\Theta(x) = e^{i\sigma x}$, $\sigma > 0$ (Theorem 2.6). Before that, we need some auxiliary constructions.

The papers [Bl1, Bl2] show that conditions sufficient for admissibility are very often expressed in terms of the Hilbert transform $\widetilde{\Omega}$ rather than Ω itself or ω. The Hilbert transform is understood in the following way:

$$h(\Omega)(x) = \widetilde{\Omega}(x) = \frac{1}{\pi} \, \mathrm{P.V.} \int_{\mathbb{R}} \Omega(t) \Big(\frac{1}{x-t} + \frac{t}{t^2+1} \Big) \, dt, \quad x \in \mathbb{R}.$$

The function $\widetilde{\Omega}$ is defined a.e. on $\mathbb{R}$ for every $\Omega \in L^1(\mathbf{P})$ (or, which is the same, for every ω with $\mathcal{L}(\omega) > -\infty$). We also need the following version of the Hilbert transformation:

$$h_0(\Omega)(x) = \frac{1}{\pi} \, \mathrm{P.V.} \int_{\mathbb{R}} \frac{\Omega(t)}{x-t} \, dt.$$

If $\Omega \in L^1(\mathbb{R})$, then $\widetilde{\Omega} = h_0(\Omega) + \mathrm{const}$.

A principal result in this paper (Theorem 2.6) is a "two-sided" refinement of the following statement due to Havin and Mashreghi and pertaining to $\mathrm{Adm}(e^{i\sigma z})$ (rather than to $\mathrm{sAdm}(e^{i\sigma z})$); see [HM2].

Theorem 1.3 (Havin–Mashreghi). *Let $\Theta(z) = e^{i\sigma z}$, $\sigma > 0$. Then any positive function ω with*

$$\text{a) } \|\widetilde{\Omega}'\|_\infty < \frac{\sigma}{2}; \qquad \text{b) } \mathcal{L}(\omega) > -\infty,$$

is Θ-admissible.

Our Theorem 2.6 shows that a) and b) ensure the property $\omega \in \mathrm{sAdm}(\Theta)$ rather than the mere inclusion $\omega \in \mathrm{Adm}(\Theta)$. Strict admissibility in the classical case was studied in the unpublished paper [BB], the results of which cover Theorem 2.6 under the additional assumption $\Omega(x) = o(x), |x| \to +\infty$. In [BB], the proof of Theorem 2.6 involves approximation of a subharmonic function by functions of the form $\log|g|$, where g is an entire function. It seems that the way to (1.2) presented here is shorter. Moreover, our method applies to a much wider class of meromorphic functions Θ. It should be noted that in [BB] it was proved that Theorem 2.6 is sharp. In that paper, for every $\sigma > 0$ a function Ω was constructed such that Ω and $\widetilde{\Omega}$ are Lipschitz, $\sigma = \|\widetilde{\Omega'}\|_\infty$, but $\Omega \notin \mathrm{sADM}(e^{iaz})$ for every $a \in (0, 2\|\widetilde{\Omega}\|_\infty)$ (however, $\Omega \in \mathrm{ADM}(e^{iaz})$ for every $a > 0$). Another proof of Theorem 2.6 is a consequence of the results of [LS] (that paper dealt with weighted Paley–Wiener spaces). I do not know if the methods of [LS] make it possible to cover other meromorphic functions Θ, as in Theorem 2.4.

§2. Main results

We denote by $|K_\Theta|$ the set of all functions of the form $|f|$, where $f \in K_\Theta$. The following description of $|K_\Theta|$ was established in [HM1].

Theorem 2.1. *Let Θ be an inner function, and let nonnegative functions m and $\omega = e^{-\Omega}$ satisfy $\mathcal{L}(\omega) > -\infty$, $\mathcal{L}(m) > -\infty$, and $m\omega \in L^2(\mathbb{R})$. Then $m\omega \in |K_\Theta|$ if and only if there exists an inner function I and an integer-valued function n such that*

$$\arg\Theta + 2\widetilde{\Omega} = 2\widetilde{\log m} + 2\pi n + \arg I. \tag{2.1}$$

Thus, if we solve equation (2.1) with the "unknowns" m, n, and I, and it turns out that $m \asymp 1$, a function $f \in K_\Theta$ with $|f| \asymp \omega$ will be found. It should be noted that $\inf m = 0$ if n is nonconstant. We will try to resolve equation (2.1) with $n = 0$. Putting $\Phi = \arg\Theta + 2\widetilde{\Omega}$, we rewrite (2.1) in the form

$$\Phi - \arg I = 2\widetilde{\log m}. \tag{2.2}$$

So, if we find an inner function I and a bounded function m that is also bounded away from zero and satisfies $\widetilde{\log m} = \Phi - \arg I$, we will prove that ω is strongly admissible.

To state the main result of the paper, we need two definitions.

Definition 2.2. A partition of the real line into intervals $J_k = [d_k, d_{k+1}]$ (where $\{d_k\}$ is a strictly monotone increasing two-sided sequence) is said to be *regular* if

$$\sup_{k\in\mathbb{Z}} \sum_{|k-l|>1} \frac{|J_l|^2}{\mathrm{dist}^2(J_k, J_l)} < +\infty.$$

Here $|J_k|$ stands for the length of the interval J_k, and $\mathrm{dist}(J_k, J_l)$ is the distance between J_k and J_l. For a regular partition, we have $\frac{|J_k|}{|J_{k+1}|} \asymp 1$. On the other hand, a partition is regular whenever $|J_k| \asymp 1$. It will be shown further that there exist regular partitions with $\inf |J_k| = 0$ (see Corollary 5.4).

Definition 2.3. A monotone increasing function Φ is said to be *regular* if there exists a sequence $\{d_k\}$ with $\Phi(d_k) = 2\pi k, k \in \mathbb{Z}$, such that the partition $J_k = [d_k, d_{k+1}]$ is regular and $\sup_{|\Phi(x)-\Phi(y)|<1} \frac{\Phi'(x)}{\Phi'(y)} < +\infty$.

Now we are ready to state the main result.

Theorem 2.4. *Suppose a meromorphic inner function* Θ *and a positive function* $\omega = e^{-\Omega} \in L^2(\mathbb{R})$ *are such that*
a) $\Omega \in L^1(\mathbf{P})$;
b) $\widetilde{\Omega} \in C^2(\mathbb{R})$;
c) *the function* $\arg\Theta + 2\widetilde{\Omega}$ *is regular.*
Then $\Omega \in \mathrm{sADM}(\Theta)$.

Before proving Theorem 2.4, we deduce from it two results pertaining to the classical case.

Theorem 2.5. *Suppose a meromorphic inner function* Θ *and a positive function* $\omega = e^{-\Omega} \in L^2(\mathbb{R})$ *satisfy*
a) $\Omega \in L^1(\mathbf{P})$;
b) $\widetilde{\Omega} \in C^1(\mathbb{R})$;
c) $0 < \inf_{\mathbb{R}}((\arg\Theta + 2\widetilde{\Omega})') \le \sup_{\mathbb{R}}((\arg\Theta + 2\widetilde{\Omega})') < +\infty$.
Then $\Omega \in \mathrm{sADM}(\Theta)$.

Proof. The function $\arg\Theta + 2\widetilde{\Omega}$ is monotone increasing. There is a sequence $\{d_k\}$ with $(\arg\Theta + 2\widetilde{\Omega})(d_k) = 2\pi k$, $k \in \mathbb{Z}$; moreover, $d_{k+1} - d_k \asymp 1$. Consequently, the partition $\{J_k\}$, where $J_k = [d_k, d_{k+1}]$, is regular, and the function $\arg\Theta + 2\widetilde{\Omega}$ is also regular. $\square$

The following statement is an immediate consequence of Theorem 2.5.

Theorem 2.6. *Let* $\Theta(z) = e^{i\sigma z}$. *If a function* $\omega = e^{-\Omega} \in L^2(\mathbb{R})$ *satisfies*
a) $\Omega \in L^1(\mathbf{P})$, *and*
b) $\|\widetilde{\Omega'}\|_\infty < \frac{\sigma}{2}$,
then $\Omega \in \mathrm{sADM}(\Theta)$.

In some special cases, it turns out to be possible to get rid of the Hilbert transformation and to obtain sufficient conditions for strong admissibility in terms of the function Ω itself. For instance, as was shown in [BB], if $\Omega \in C^2(\mathbb{R}) \cap L^1(\mathbf{P})$ monotonically increases on the ray $[A, +\infty)$ and monotonically decreases on $(-\infty, A]$, then $\Omega \in \mathrm{sADM}(e^{i\sigma z})$ for every $\sigma > 0$. The results of [Bl1, Theorem 9] show that if

$$\Omega \in L^1(\mathbf{P}), \quad \|\Omega'\|_\infty < +\infty, \quad \text{and} \quad \Delta_2(\Omega) \in L^1(\mathbf{P}),$$

where

$$\Delta_2(\Omega)(t) = \sup_{x\in\mathbb{R}} |\Omega(x+t) - 2\Omega(x) + \Omega(x-t)|,$$

then $\|\widetilde{\Omega'}\|_\infty < +\infty$. This *implies the strong admissibility* of ω for $K_{e^{i\sigma z}}$ with sufficiently large σ. Theorem 2.5 is applicable not only when $(\arg\Theta)' \asymp 1$ (as in the classical case). For example, if B is a Blaschke product with zeros $z_k = x_k + iy_k$ ($k \in \mathbb{Z}$) and $\inf y_k = 0$, then there is some hope to improve the behavior of $\arg\Theta$ with the help of $2\widetilde{\Omega}$. For instance, combining a result of [Bl3] (Corollary 6.1) and Theorem 2.5, we arrive at the following statement. Put

$$A_B = \sum_{k\in\mathbb{Z}} \log\Big(1 + \frac{1 - y_k^2}{(x - x_k)^2 + y_k^2}\Big).$$

Corollary 2.7. *Let* B *be a Blaschke product with zeros* $z_k = x_k + iy_k$, $k \in \mathbb{Z}$. *If a function* $\omega = e^{-\Omega} \in L^2(\mathbb{R})$ *is such that* Ω *belongs to* $L^1(\mathbf{P})$ *and is representable in the form* $A_B + \Omega_1$, *where* $\widetilde{\Omega_1}$ *is Lipschitz with a sufficiently small Lipschitz constant, then* $\Omega \in \mathrm{sADM}(B)$.

§3. Outline of the proof of Theorem 2.4: Preliminary remarks

In order to resolve equation (2.1), we formally rewrite it by using the Hilbert transformation. This suggests the following definition of m:

$$2\log m = (\arg I - \Phi)^{\sim},$$

where I is an appropriate inner function that ensures the boundedness of $\log m$. (Strictly speaking, the definition of $\log m$ will not be so straightforward; here we only outline the principal idea.) The choice of I is suggested by the rate of growth of Φ (see condition a) in Theorem 2.4). We define I to be the Blaschke product with zeros $z_k = x_k + iy_k$, where a monotone increasing sequence x_k behaves in the same way as the numbers d_k determined by $\widetilde{\Phi}(d_k) = 2\pi k$. We must specify the choice of x_k and y_k so as to ensure the boundedness of the right-hand side of (2.2) on $\mathbb{R}$. Some preliminary remarks are required for that (recall that $h(f)$ and $\widetilde{f}$ denote the Hilbert transform with the regularized Cauchy kernel, and $h_0(f)$ is the usual Hilbert transform).

Lemma 3.1. *Suppose $f \in L^1(a,b)$, $f \equiv 0$ off $[a,b]$, and $\int_{\mathbb{R}} f(s)\,ds = 0$. Then*

$$|h_0(f)(x)| \le \frac{1}{\pi}\frac{\int_a^b\int_a^t |f(s)|\,ds\,dt}{\operatorname{dist}^2(x,[a,b])}, \quad x \notin [a,b].$$

Proof. Let F be the primitive for f with $F(a) = 0$. Then $F(b) = 0$. If $x \in [a,b]$, then

$$|h_0(f)(x)| = \frac{1}{\pi}\left|\int_a^b \frac{f(t)}{x-t}\,dt\right| = \frac{1}{\pi}\left|\int_a^b F(t)\frac{dt}{(x-t)^2}\right| \le \frac{1}{\pi}\cdot\frac{\int_a^b |F(t)|\,dt}{\operatorname{dist}^2(x,[a,b])}. \qquad \square$$

Lemma 3.2. *If f is as in Lemma 3.1, then*

$$|h_1(f)(x)| \le |h_0(f)(x)| + \int_a^b \frac{|\int_a^t f(s)ds|}{t^2+1}\,dt, \quad x \in \mathbb{R}.$$

Proof. It is easily seen that

$$|h(f)(x) - h_0(f)(x)| = \left|\int_a^b f(t)\frac{t}{t^2+1}\,dt\right| = \left|\int_a^b\int_a^t f(s)\,ds\Big(\frac{t}{t^2+1}\Big)'\,dt\right|,$$

and $\left|\left(\frac{t}{t^2+1}\right)'\right| \le \frac{1}{t^2+1}$. $\square$

Here and in what follows, we define I to be a meromorphic Blaschke product with zeros $z_k = x_k + iy_k$ $(x_k \in \mathbb{R},\ y_k > 0,\ k \in \mathbb{Z})$, where the sequence $\{x_k\}_{k\in\mathbb{Z}}$ is monotone increasing. It is known that

$$(\arg I)'(x) = \sum_k \frac{2y_k}{(x-x_k)^2 + y_k^2}, \quad x \in \mathbb{R}$$

(see [HM2, p. 1259]). Under the condition

$$\sum_{k\in\mathbb{Z}} \frac{y_k + y_k^2}{x_k^2} < +\infty,$$

we associate with I a positive function R_I on $\mathbb{R}$ defined by the formula

$$R_I(x) = \sum_k \log\Big(1 + \frac{y_k^2}{(x-x_k)^2}\Big), \quad x \in \mathbb{R}. \tag{3.1}$$

Lemma 3.3. *The function R_I is integrable with respect to the Poisson measure (i.e., $R_I \in L^1(\mathbf{P})$) and*

$$-h(R_I) = \arg I - 2\pi n_I + \text{const}, \tag{3.2}$$

where n_I is the counting function for the function x_k, that is, $n_I(t) = \operatorname{card}\{k : 0 \le x_k < t\}$ for $t \ge 0$, and $n_I(t) = -\operatorname{card}\{k : t < x_k < 0\}$ for $t < 0$.

Proof. The series (3.1) converges for $x \neq x_k$ because $\log\bigl(1 + \frac{y_k^2}{(x-x_k)^2}\bigr) \le \frac{y_k^2}{(x-x_k)^2}$ and the series $\sum_k \frac{y_k^2}{x_k^2}$ converges. In order to prove that $R_I \in L^1(\mathbf{P})$, we observe that

$$\pi \int_{\mathbb{R}} \log((x-X)^2 + Y^2)\, d\mathbf{P}(x) = \pi \log(X^2 + (1+Y)^2), \quad Y \ge 0. \tag{3.3}$$

Indeed, let $Z := X - iY$. By the residue theorem,

$$\begin{aligned}
\int_{\mathbb{R}} \frac{\log((x-X)^2+Y^2)}{1+x^2}\, dx &= 2\operatorname{Re} \int_{\mathbb{R}} \frac{\log(x-Z)}{1+x^2}\, dx \\
&= 2\operatorname{Re}\Bigl[2\pi \cdot i \operatorname{res}_i \Bigl(\frac{\log(z-Z)}{1+z^2}\Bigr)\Bigr] \\
&= 2\operatorname{Re}[\pi i \log((z-i)/i)] = 2\pi \log|X + i(Y+1)| \\
&= \pi \log(X^2 + (1+Y)^2).
\end{aligned}$$

Letting Y tend to zero, we see that the result remains true also for $Y = 0$. Now, (3.3) with $X = x_k$ and $Y = y_k$ or $Y = 0$ implies

$$\int_{\mathbb{R}} \log \frac{(x-x_k)^2 + y_k^2}{(x-x_k)^2}\, d\mathbf{P}(x) = \log \frac{x_k^2 + (1+y_k)^2}{x_k^2+1} \le \frac{2y_k + y_k^2}{x_k^2+1}.$$

Since the series $\sum_k \frac{y_k + y_k^2}{x_k^2}$ converges, we conclude that $R_I \in L^1(\mathbf{P})$.

Instead of (3.2), we verify the formula

$$-h(R_I) + 2\pi n_I = \arg I + c. \tag{3.4}$$

For this, we show that

a) the left-hand side of (3.4) can be defined at the points x_k to become continuous;

b) off the x_k, the left-hand side is differentiable and its derivative coincides with $(\arg I)'$, which implies (3.4) and (3.2).

a) The series defining R_I converges in $L^1(\mathbf{P})$; consequently,

$$h(R_I)(x) = \sum_{k\in\mathbb{Z}} h\Bigl(\log \frac{(x-x_k)^2 + y_k^2}{(x-x_k)^2}\Bigr), \quad x \in \mathbb{R}.$$

It is easily seen that the last series converges uniformly on every compact interval separated away from the x_k. We fix $l \in \mathbb{Z}$. On (x_{l-1}, x_{l+1}), we have

$$-h(R_I) = h(\log(x-x_l)^2) + r_l, \tag{3.5}$$

where the function r_l is continuous on (x_{l-1}, x_{l+1}). Note that

$$h(2\log|x - x_l|) = 2\arg(t - x_l) + \text{const}, \quad t \neq x_l \tag{3.6}$$

($\arg u := 0$ for $u > 0$, and $\arg u := \pi$ for $u < 0$). From (3.5) and (3.6) we deduce that the function $-h(R_I)(t)$ has a first kind discontinuity (jumps down) when t passes through x_l, whereas $2\pi n_I$ jumps up (by $+2\pi$). Consequently, the limits from the left and from the right of $-h(R_I) + 2\pi n_I$ at x_l coincide.

b) Formal differentiation of the series defining $h(R_I)$ yields a series converging uniformly on every compact interval free of points x_k (this is a direct consequence of the fact that the series defining R_I converges in $L^1(\mathbf{P})$). Therefore, termwise differentiation on (x_l, x_{l+1}), which annihilates n_I on (x_l, x_{l+1}), yields

$$\begin{aligned}(-h(R_I)+2\pi n_I)'(x) &= \sum_k -\big(h_0(\log((x-x_k)^2+y_k^2)) - h_0(\log(x-x_k)^2)\big)' \\ &= \sum_k -h_0([\log((x-x_k)^2+y_k^2)]') = -\sum_k h\Big(\frac{2(x-x_k)^2}{(x-x_k^2)+y_k^2}\Big) \\ &= \sum_k \frac{2y_k}{(x-x_k)^2+y_k^2} = (\arg I)'(x), \quad t \neq x_k, \quad k \in \mathbb{Z}.\end{aligned}$$

We observe that the function $h_0(\log(x-x_k)^2)$ is piecewise constant and $h_0\big(\frac{x}{1+x^2}\big) = -\frac{1}{1+x^2}$. □

§4. Proof of Theorem 2.4

By assumption, the function $\Phi = \arg\Theta + 2\widetilde{\Omega}$ is strictly monotone increasing on the real axis. Let d_k satisfy $\Phi(d_k) = 2\pi k$, $k \in \mathbb{Z}$. We put $J_k = [d_k, d_{k+1}]$ and denote $a_k = \frac{1}{|J_k|}\int_{J_k} \Phi \in (2\pi k, 2\pi(k+1))$, where $a_k = 2\pi(k+q_k)$, $q_k \in (0,1)$. Now we define $x_k = d_{k+1} - q_k|J_k|$, so that x_k is an inner point of J_k. Let n_I be the counting sequence for $\{x_k\}$, and I the Blaschke product whose zeros are $x_k + iy_k$. There is no loss of generality in assuming that $d_0 < 0 < x_0 < d_1$ (consequently, $n_I(d_k) = k$). Put

$$\Psi = \Phi - 2\pi n_I.$$

Clearly,

$$|\Psi| \le 2\pi, \qquad \int_{J_k} \Psi = 0 \quad \text{for every} \quad k \in \mathbb{Z}.$$

Indeed,

$$\begin{aligned}\int_{J_k} \Psi &= |J_k|a_k - 2\pi\int_{J_k} n_I |J_k|a_k - 2\pi k|J_k| + 2\pi(d_{k+1}-x_k) \\ &= 2\pi|J_k|q_k - 2\pi d_{k+1} + 2\pi x_k = 0.\end{aligned}$$

Note that $\Psi(d_k) = 0$. Let

$$2\log m = R_I - \widetilde{\Psi}. \tag{4.1}$$

When the boundedness of $\log m$ is proved, the inclusion $\widetilde{\Psi} \in L^1(\mathbf{P})$ will follow from (4.1) because $R_I \in L^1(\mathbf{P})$ (see Lemma 3.3). Then we will be able to apply h to the two sides of (4.1), obtaining

$$\widetilde{2\log m} = \widetilde{R_I} + \Psi + \text{const} = -\arg I + 2\pi n_I + \Phi - 2\pi n_I + \text{const},$$

by Lemma 3.3. Since $\Phi = \arg\Theta + 2\widetilde{\Omega}$, it will follow that

$$\widetilde{2\log m} + \arg I = \arg\Theta + 2\widetilde{\Omega} + \text{const}.$$

Since m is bounded and separated away from zero, we conclude that ω is strongly admissible (see §2).

It remains to prove that $\log m$ is a bounded function. Fixing $k \in \mathbb{Z}$, we put

$$J = J_{k-1} \cup J_k \cup J_{k+1}.$$

Also, we define

$$R_I^s(t) = \log\frac{1+(t-x_s)^2}{(t-x_s)^2}, \quad \Psi_s = \Psi \cdot \chi_{J_s}, \quad t \in \mathbb{R}, \; s \in \mathbb{Z}.$$

Now, formula (4.1) turns into

$$\begin{aligned} 2\log m &= \sum_{|s-k|>1} R_I^s - \sum_{|s-k|>1} h(\Psi_s) \\ &\quad + \Big\{[R_I^{k-1} + R_I^k + R_I^{k+1}] - h[\Psi_{k-1} + \Psi_k + \Psi_{k+1}]\Big\} \\ &= V_1 - V_2 + V_3. \end{aligned}$$

We estimate V_1:

$$\begin{aligned} 0 < \sum_{|s-k|>1} R_I^s &= \sum_{|k-s|>1} \log\Big(1 + \frac{y_s^2}{(x-x_s)^2}\Big) \\ &\le \sum_{|k-s|>1} \frac{y_s^2}{(x-x_s)^2} \le \sum_{|k-s|>1} \frac{|J_s|^2}{\operatorname{dist}^2(J_k, J_s)}. \end{aligned}$$

Since the partition J_k is regular, the last sum is bounded uniformly in k. Now, we estimate V_2. Note that $\int_{J_s} \Psi_s = 0$. Therefore, $\int_{J_s}\int_{J_s} |\Psi(s)|\,ds\,dt \le 2\pi|J_s|^2$. So, using Lemmas 3.1 and 3.2, we can estimate $|h(\Psi)|$:

$$|h(\Psi_s)(x)| \le C_1 \frac{|J_s|^2}{\operatorname{dist}^2(J_k, J_s)} + C_2 \int_{J_s} \frac{dt}{t^2+1}.$$

Summing over all s with $|s-k| > 1$ and using the fact that the partition J_k is regular, we easily deduce that V_2 is uniformly bounded.

We pass to the subtlest estimate, namely, to that of V_3. Put

$$\Upsilon = \Psi_{k-1} + \Psi_k + \Psi_{k+1} + 2\pi(\chi_{(x_{k-1},d_{k+2})} + \chi_{(x_k,d_{k+2})} + \chi_{(x_{k+1},d_{k+2})}).$$

Thus,

$$V_3 = R_I^{k-1} + R_I^k + R_I^{k+1} + 2\pi h(\chi_{(x_{k-1},d_{k+2})} + \chi_{(x_k,d_{k+2})} + \chi_{(x_{k+1},d_{k+2})}) - h(\Upsilon).$$

We show that the function $h(\Upsilon)$ is bounded. The function Υ itself is continuous and differentiable on (d_{k-1}, d_{k+2}). Also, we emphasize that $\|\Upsilon\|_\infty \le 6\pi$ and $|\Upsilon'(x)| \le \frac{C}{|J_k|}$, $x \in J$, where C does not depend on k because $\frac{1}{|J_k|} = \Upsilon'(\xi) = \Phi'(\xi)$ for $\xi \in J_k$. Put $m_k = \min\{|J_{k-1}|, |J_k|, |J_{k+1}|\}$, $M_k = \max\{|J_{k-1}|, |J_k|, |J_{k+1}|\}$. The partition J_k is regular; consequently, $M_k \asymp m_k$. Furthermore,

$$\begin{aligned} h_0(\Upsilon)(x) &= \text{P.V.}\int_J \frac{\Upsilon(t)}{x-t}\,dt \\ &= \text{P.V.}\int_{|x-t|\le m_k} \frac{\Upsilon(t)-\Upsilon(x)}{x-t}\,dt + \int_{|x-t|>m_k, x\in J} \frac{\Upsilon(t)}{x-t}\,dt. \end{aligned}$$

The absolute value of the first integral does not exceed $\frac{2Cm_k}{|J_k|}$, and the modulus of the second integral does not exceed $\|\Upsilon\|_\infty \cdot 2\log\frac{2M_k}{m_k}$. It can easily be shown that

$$|h(\Upsilon)(x) - h_0(\Upsilon)(x)| \le \|\Upsilon\|_\infty \cdot \int_J \frac{|t|}{t^2+1}\,dt.$$

(The last integral is bounded because the system J_k is regular.) Consequently, the function $h(\Upsilon)$ is bounded on J_k by a constant depending on k. It only remains to estimate the quantity

$$R_I^{k-1} + R_I^k + R_I^{k+1} + 2\pi h(\chi_{(x_{k-1},d_{k+2})} + \chi_{(x_k,d_{k+2})} + \chi_{(x_{k+1},d_{k+2})}).$$

We can use the operator h_0 in place of h in this expression (because the difference is a bounded function), so that the expression in question turns into

$$\begin{aligned}\Big[\sum_{|s-k|\le 1}\log((x-x_s)^2+y_s^2)\Big]-6\log|x-d_{k+2}|\\ =\sum_{|s-k|\le 1}\Big[\log((x-x_s)^2+y_s^2)-2\log|x-d_{k+2}|\Big].\end{aligned}\tag{4.2}$$

We treat the quantity $-6\log|x-d_{k+2}|$ as the sum of three summands $-2\log|x-d_{k+2}|$ and distribute them so as to compensate each of the terms $\log((x-x_s)^2+y_s^2)$. Observe that

$$\frac{|J_s|^2}{2M_k^2}\le\frac{(x-x_s)^2+y_s^2}{(x-d_{k+2})^2}\le\frac{4M_k^2+|J_s|^2}{|J_s|^2}$$

for $s=k-1,k,k+1$. Therefore, each summand in the last sum in (4.2) is bounded, and the theorem is proved. □

§5. Regularity condition: Examples of application of Theorem 2.4

In this section, we discuss regularity conditions and show some applications of Theorem 2.4. First, we observe that regularity does not impose any growth restrictions on a function (in the sense that a regular function may grow arbitrarily fast). On the other hand, the regularity of the partition J_k forbids *slow* growth (a regular function cannot grow more slowly than $\log^2|x|$ at infinity). We recall that the regularity condition for a strictly monotone increasing function consists of two parts:

a) the regularity of the partition $J_k=[f^{-1}(2\pi k),f^{-1}(2\pi(k+1))]$;

b) $\sup_{|f(x)-f(y)|<1}\left|\frac{f'(x)}{f'(y)}\right|$ for all sufficiently large x and y.

The following statement shows that sometimes b) is a consequence of a).

Lemma 5.1. *If $f\in C^2(\mathbb{R})$ changes convexity to concavity and vice versa finitely many times, then* b) *follows from* a).

Proof. There is no loss of generality in assuming that f is concave (i.e., $f''\ge 0$). Let $x\in J_k$. We estimate $f'(x)$:

$$f'(y)\le f'(x)\le f'(z),$$

for every $y\in J_{k-1}$, $z\in J_{k+1}$. Choose y and z in such a way that

$$\begin{aligned}f'(y)(d_k-d_{k-1})&=f'(y)|J_{k-1}|=2\pi,\\ f'(z)(d_{k+2}-d_{k+1})&=f'(z)|J_{k+1}|=2\pi.\end{aligned}$$

Since the partition J_k is regular, we have $|J_k|\asymp|J_{k-1}|\asymp|J_{k+1}|$. Consequently, if $x\in J_k$, then $f'(x)\asymp\frac{2\pi}{|J_k|}$. Thus, b) is fulfilled. □

For example, if Θ is the Blaschke product with zeros iy_k, $y_k>0$, then $\arg\Theta$ has one convexity-concavity change. Indeed,

$$(\arg\Theta)''(x)=-2x\sum_k\frac{y_k}{(x^2+y_k^2)^2}.$$

In Theorem 2.4, it is assumed that the function $\arg\Theta+2\widetilde{\Omega}$ is regular. As will be shown further, for many regular functions $\arg\Theta$ there are fairly many functions $\widetilde{\Omega}$ such that $\arg\Theta+2\widetilde{\Omega}$ is also regular.

Lemma 5.2. *Suppose f is a regular function with $f(0)=0$ and*

$$\sup_{1/2\le|x/y|\le 2}\max\Big(\frac{f(x)}{f(y)};\frac{f^{-1}(x)}{f^{-1}(y)}\Big)<+\infty,$$

and g is a smooth function with $g(0)=0$ and $|g'|\le qf'$ for some $q\in(0,1)$. Then $f+g$ is also regular.

Proof. It is easily seen that $|g|\le q|f|$. Therefore, $f+g$ is monotone increasing and $f+g\asymp f$. Let d_k and d'_k be determined by $f(d_k)=2\pi k$ and $f(d'_k)+g(d'_k)=2\pi k$. Next,

$$2\pi k=f(d'_k)+g(d'_k)=Cf(d'_k),$$

where C is bounded from above and from below by constants depending only on g. The properties of f show that $d'_k\asymp d_k$. Now, for some $x\in J_k=[d_k,d_{k+1}]$ we have

$$f'(x)(d_{k+1}-d_k)=2\pi;$$

on the other hand, for some $y\in J'_k=[d'_k,d'_{k+1}]$ we have

$$(f'(y)+g'(y))(d'_{k+1}-d'_k)=2\pi=Cf'(y)(d'_{k+1}-d'_k).$$

Consequently, $|J_k|=d_{k+1}-d_k\asymp|J'_k|=d'_{k+1}-d'_k$, and

$$\operatorname{dist}(J_k,J_l)\asymp\operatorname{dist}(J'_k,J'_l).$$

Therefore, the partition J'_k is also regular; with it, $f+g$ is regular. □

Lemma 5.2 allows us to give several applications of Theorem 2.4.

Corollary 5.3. *Let functions Θ and ω satisfy $\arg\Theta+2\widetilde{\Omega}\asymp(1+|x|)^\beta$ and $(\arg\Theta+2\widetilde{\Omega})'\asymp(1+|x|)^{\beta-1}$ $(1\le\beta<2)$. Then $\Omega\in\mathrm{sADM}(\Theta)$.*

Proof. Consider the partition $J_k=[d_k,d_{k+1}]$. It is easily seen that

$$|d_k|\asymp(1+|k|)^{\frac{1}{\beta}},\quad |J_k|\asymp(1+|k|)^{\frac{1}{\beta}-1}.$$

Then the function $\arg\Theta+2\widetilde{\Omega}$ is regular. Indeed, take $s=(1/\beta)-1\in(-1/2,0)$. We know that $|J_k|\asymp|k|^s\le 1$ and $|d_k|\asymp|k|^{s+1}$ for $k\ne 0$. Without loss of generality, we may assume that $0\in J_0$. If $k>0$, then

$$\sum_{|l-k|>1}\frac{|J_l|^2}{\operatorname{dist}^2(J_k,J_l)}=\sum_{l\le 0}+\sum_{0<l<k/2}+\sum_{k/2\le l<k-1}+\sum_{l>k+1}=S_1+S_2+S_3+S_4. \tag{5.1}$$

First, we observe that $S_1\le\mathrm{const}\sum_{l\le 0}(d_l)^{-2}$, and the last sum is bounded uniformly in k. Second, if $l>k>0$, we have $\operatorname{dist}(J_l,J_k)\ge\mathrm{const}\,|J_l|\cdot|l-(k+1)|$. Consequently,

$$S_4\le\mathrm{const}\sum_{l>k+1}\frac{1}{|l-(k+1)|^2}.$$

We note that for $0\le l<k-1$ we have

$$\operatorname{dist}(J_k,J_l)=\sum_{n=l+1}^{k-1}|J_n|\asymp\sum_{n=l+1}^{k-1}|n|^s\asymp|k|^{s+1}-|l|^{s+1}.$$

Therefore,

$$
\begin{aligned}
S_2 &\le \text{const} \sum_{0<l<k/2} \Big(\frac{|k|^s}{|k|^{s+1} - |l|^{s+1}} \Big)^2 \\
&\le \text{const} \sum_{0<l<k/2} \frac{1}{|k|^{2(s+1)}} \le \text{const}\, |k|^{-2s-1} \le \text{const}, \qquad (5.2)\\
S_3 &\le \text{const} \sum_{k/2 \le l < k-1} \Big(\frac{|k|^s}{|k|^{s+1} - |l|^{s+1}} \Big)^2 \le \text{const} \sum_{l<k-1} \frac{1}{(k-l)^2}.
\end{aligned}
$$

So, we have estimated the quantities S_1, S_2, S_3, and S_4 on the right of (5.1). For $k < 0$ the proof is similar. □

The above assumption allows us to prove that certain specific majorants are admissible. Here is an example.

Corollary 5.4. *Let B_α denote the Blaschke product with the zeros $z_k = \operatorname{sgn}(k)|k|^\alpha + i$, $k \in \mathbb{Z}$, $1/2 < \alpha < 1$. If a function $\omega = e^{-\Omega} \in L^2(\mathbb{R})$ satisfies*
a) $\Omega \in L^1(\mathbf{P})$, *and*
b) $|\widetilde{\Omega}'(x)| < C|x|^{1/\alpha - 1}$ *for some* $C < \frac{\pi}{\alpha}$,
then $\Omega \in \mathrm{sADM}(B_\alpha)$.

Proof. This is a direct consequence of Corollary 5.3 and the estimate $(\arg B_\alpha)'(x) = \frac{2\pi}{\alpha}|x|^{1/\alpha-1} + O(1)$ (see [HM2, p. 1298]). □

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