Group theory and homotopy groups of spheres. Wu's formula

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$\pi(H_1, H_2)$

- Let G be a group and $H_1, H_2 \lhd G$.
- $[H_1, H_2] = \langle\{[h_1, h_2] \mid h_1 \in H_1, h_2 \in H_2\}\rangle$
- $[h_1, h_2] = h_1^{-1}h_2^{-1}h_1h_2$
- $[H_1, H_2] \subseteq H_1 \cap H_2$.
- $$\pi(H_1, H_2) := \frac{H_1 \cap H_2}{[H_1, H_2]}$$
- $\pi(H_1, H_2)$ is abelian.
- **Example**: $\pi(H, H) = H_{\mathsf{ab}}$.
- **Example**: If $G = F(x, y)$ is the free group, $H_1 = \langle x\rangle^G$, $H_2 = \langle y\rangle^G$, then $\pi(H_1, H_2) = 0$.
- Informally $\pi(H_1, H_2)$ measures how much H_1 and H_2 are 'linked'.

$\pi(H_1, H_2, H_3)$

- $H_1, H_2, H_3 \lhd G$.
- $[\![H_1, H_2, H_3]\!] = [H_1 \cap H_2, H_3] \cdot [H_1 \cap H_3, H_2] \cdot [H_2 \cap H_3, H_1]$ **fat commutator** (non-standard term).
- $[H_1, H_2, H_3]_S = [[H_1, H_2], H_3] \cdot [[H_1, H_3], H_2] \cdot [[H_2, H_3], H_1]$ **symmetric commutator**.
- $[H_1, H_2, H_3]_S \subseteq [\![H_1, H_2, H_3]\!]$
- $$\pi(H_1, H_2, H_3) := \frac{H_1 \cap H_2 \cap H_3}{[\![H_1, H_2, H_3]\!]}$$
- $\pi(H_1, H_2, H_3)$ is abelian.

$\pi(H_1,\dots,H_n)$

- $$[\![H_1,\dots,H_n]\!] = \prod_{I\sqcup J=\{1,\dots,n\}}\left[\bigcap_{i\in I} H_i\ ,\ \bigcap_{j\in J} H_j\right]$$
  fat commutator (non-standard term).
- $$[H_1,\dots,H_n]_S = \prod_{\sigma\in\Sigma_n}[H_{\sigma(1)},\dots,H_{\sigma(n)}]$$
  symmetric commutator.
- $[H_1,\dots,H_n]_S \subseteq [\![H_1,\dots,H_n]\!]$
- $$\pi(H_1,\dots,H_n) := \frac{H_1\cap\dots\cap H_n}{[\![H_1,\dots,H_n]\!]}$$
  is abelian.
- Informally $\pi(H_1,\dots,H_n)$ measures how much $H_1,\dots,H_n$ are 'linked'.

Wu's formula

- $G := F(x_0, \dots, x_{n-1})$
- $R_i := \langle x_i \rangle^G \quad$ for $0 \le i \le n-1$
- $R_n := \langle x_0 x_1 \dots x_{n-1} \rangle^G$
- **Theorem.** For $n \ge 2$

$$\pi(R_0, \dots, R_n) = \pi_{n+1}(S^2)$$

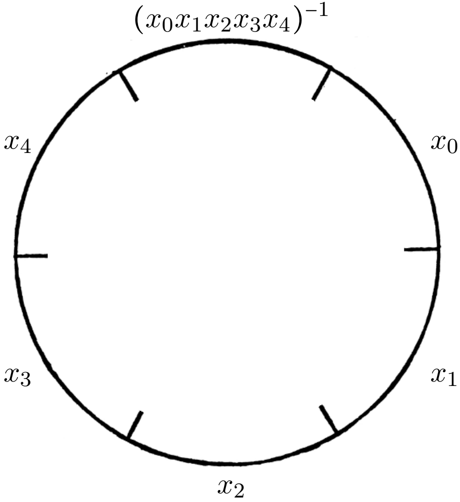
- **Lemma.** $[\![R_0, \dots, R_n]\!] = [R_0, \dots, R_n]_S$
- **Wu's formula**: For $n \ge 2$

$$\pi_{n+1}(S^2) = \frac{R_0 \cap \dots \cap R_n}{[R_0, \dots, R_n]_S}$$

- **Corollary.** $\pi_{n+1}(S^2) = Z(F/[R_0, \dots, R_n]_S)$

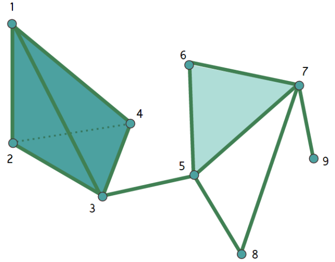
Informally: the groups $R_0, \ldots, R_n$ are 'linked'

Informally: the groups $\langle x_0 \rangle^F, \ldots, \langle x_{n-1} \rangle^F, \langle x_0 \ldots x_{n-1} \rangle^F$ are 'linked'.



The product of elements in the circle is trivial.

Simplicial homotopy theory. Abstract simplicial complexes

- An **abstract simplicial complex** K is a family of finite sets such that if $X \in K$ and $Y \subseteq X$ then $Y \in K$.
- 

$K = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\}$
$\{1,2\},\{1,4\},\{1,3\},\{2,3\}\{2,4\},\{3,4\}\{3,5\},$
$\{5,6\},\{5,7\},\{5,8\},\{6,7\},\{7,8\},\{7,9\},$
$\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{5,6,7\},$
$\{1,2,3,4\}\}$

- $\mathsf{Vert}(K) = \bigcup_{X \in K} X$
- n**-simplex** of K is a $n+1$-element set in K
- There is a natural way to associate topological space called geometric realisation

$$K \mapsto |K|.$$

- An abstract simplicial complex is a good and simple combinatorial model for a space.
- We can define morphisms of abstract simplicial complexes.
- But we **can not** define **homotopy** of such morphisms in a combinatorial way.
- If we want to develop homotopy theory in a combinatorial way, we should use a less intuitive notion of **simplicial set**.
- It is less intuitive because simplicial sets contain **degenerate simplexes** whose geometric interpretation is not obvious.

Simplicial sets

A **simplicial set** X is a sequence of sets

$$X_0, X_1, \ldots$$

together with maps $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ for $0 \le i \le n$ satisfying identities:

1. $d_i d_j = d_{j-1} d_i$ if $i < j$;
2. $s_i s_j = s_j s_{i-1}$ if $i > j$;
3. $d_i s_j = s_{j-1} d_i$ if $i < j$;
4. $d_i s_i = \mathsf{id} = d_i s_{i+1}$;
5. $d_i s_j = s_j d_{i-1}$ if $i > j + 1$.

d_i is called ith **face**. s_i is called ith **degeneracy**.
Elements of X_n are called n-simplexes.
An n-simplex is called **degenerate** if $x = s_i(y)$ for some i and y.
Again there is a notion of geometric realisation

$$X \mapsto |X|.$$

Example: simplicial set of an abstract simplicial complex

- Let K be an abstract simplicial complex. Assume that $\mathsf{Vert}(K)$ is totally ordered.
- $S(K)$ is a simplicial set consisting of **ordered** tuples of vertices:

$$S(K)_n = \{(v_0, \dots, v_n) \mid v_0 \le \dots \le v_n, \{v_0, \dots, v_n\} \in K\}$$
$$d_i(v_0, \dots, v_n) = (v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$$
$$s_i(v_0, \dots, v_n) = (v_0, \dots, v_i, v_i, \dots, v_n)$$

- $(v_0, \dots, v_n)$ is degenerate if $v_i = v_{i+1}$ for some i.
- $|K| = |S(K)|$

Homotopy groups of siplicial sets

- A pointed simplicial set X is a simplicial set such that each set X_n is pointed $* \in X_n$ and d_i, s_i preserve the base points.
- If X is a pointed simplicial set, then $|X|$ is pointed.
- If X is a pointed siplicial set we define homotopy groups in the 'stupid' way:

$$\pi_i(X) := \pi_i(|X|).$$

- There is an 'internal' definition that does not use topological spaces. But it requires more theory.
- Moreover, all algebraic topology can be developed in internal terms of simplicial sets.

Homotopy groups of simplicial groups. Moore complex

- **Simplicial group** is a simplicial set G whose components are groups G_n and d_i, s_i are homomorphisms.
- **Moore complex** $N(G)$ consists of (non-abelian) groups

$$N_n(G) = \bigcap_{i\neq 0} \mathrm{Ker}(d_i : G_n \to G_{n-1})$$

  and differentials

$$\partial_n^G : N_n(G) \to N_{n-1}(G), \qquad \partial_n^G(g) = d_0(g).$$

- $\mathrm{Im}(\partial_{n+1}^G) \lhd \mathrm{Ker}(\partial_n^G)$
- **Theorem**:

$$\pi_n(G) \cong \frac{\mathrm{Ker}(\partial_n^G)}{\mathrm{Im}(\partial_{n+1}^G)}.$$

- We can compute homotopy groups of simplicial groups without topology.

Homotopy groups of simplicial groups. Degenerate components

- **Theorem.** Let G be a simplicial group and G_{n+1} is generated as a group by degenerate simplexes. Set $K_i := \mathrm{Ker}(d_i : G_n \to G_{n-1})$. Then

$$\pi_n(G) = \pi(K_0, \dots, K_n)$$

- J.L. Castiglioni and M. Ladra: Peiffer elements in simplicial groups and algebras, J. Pure Appl. Alg., 212, (2008), 2115-2128.

Milnor's $F[X]$-construction for a simplicial set X

- For a set X we denote by $F(X)$ the free group generated by X.
- For a pointed set X we denote by $F[X]$ the quotient

$$F[X] = F(X)/(* = 1).$$

- $F[X] \cong F(X \setminus \{*\})$ is a free group.
- For a pointed simplicial set X we define a simplicial group $F[X]$ component-wise $F[X]_n = F[X_n]$ and homomorphisms $d_i : F[X_n] \to F[X_{n-1}]$ and $s_i : F[X_n] \to F[X_{n+1}]$ are induced by d_i, s_i for X.
- $F[X]$ is called **Milnor's construction** of X.
- **Theorem.**

$$\pi_{n+1}(\Sigma|X|) = \pi_n(F[X]),$$

  where $\Sigma|X|$ is the suspension of $|X|$.
- Hence, in order to compute homotopy groups of the suspension of a space (for example $S^2 = \Sigma S^1$) it is enough to use group theory.

Simplicial circle

- S^1 is a pointed simplicial set such that

$$(S^1)_n = \{*, x_0, \dots, x_{n-1}\}$$

$d_0(x_0) = *;$
$d_j(x_i) = x_{i-1}$ for $j \le i \ne 0;$
$d_j(x_i) = x_i$ for $j > i \ne n-1;$
$d_n(x_{n-1}) = *.$
$s_j(x_i) = x_i$ for $j > i$
$s_j(x_i) = x_{i+1}$ for $j \le i.$

- $|S^1|$ is the usual circle.

Milnor's construction of the simplicial circle. Wu's formula

- $F[S^1]_n = F(x_0, \dots, x_{n-1});$
- $K_0 := \mathrm{Ker}(d_0) = \langle x_0 \rangle^F$
- $K_i := \mathrm{Ker}(d_i) = \langle x_{i-1}^{-1} x_i \rangle^F$ for $1 \le i \le n-1$
- $K_n := \mathrm{Ker}(d_n) = \langle x_{n-1} \rangle^F$
- $$\pi_{n+1}(S^2) = \pi_n(F[S^1]) = \pi(K_0, \dots, K_n)$$
- If we change the basis

$$x'_0 = x_0, \qquad x'_i = x_{i-1}^{-1} x_i,$$

for $1 \le i \le n-1$, then for $0 \le j \le n-1$

$$K_j = \langle x'_j \rangle^F, \qquad K_n = \langle x'_0 \cdot \ldots \cdot x'_{n-1} \rangle^F.$$

- Then $\pi_{n+1}(S^2) = \pi(\langle x'_0 \rangle^F, \dots, \langle x'_{n-1} \rangle, \langle x'_0 \cdot \ldots \cdot x'_{n-1} \rangle).$

Appendix: our result

- If $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ are ideals of a ring R we set

$$\|\mathfrak{a}_1, \dots, \mathfrak{a}_n\| = \sum_{I \sqcup J = \{1,\dots,n\}} \left(\bigcap_{i \in I} \mathfrak{a}_i \right) \cdot \left(\bigcap_{j \in J} \mathfrak{a}_j \right)$$

- If $\mathfrak{a} \le \mathbb{Z}[G]$ the dimension subgroup of $\mathfrak{a}$ is

$$D(\mathfrak{a}) = G \cap (1 + \mathfrak{a}).$$

- **Theorem** (R. Mikhailov, J. Wu, –) Let R, S, T be normal subgroups of a group G. Consider ideals of $\mathbb{Z}[G]$

$$\mathbf{r} = (R-1)\mathbb{Z}[G], \quad \mathbf{s} = (S-1)\mathbb{Z}[G], \quad \mathbf{t} = (T-1)\mathbb{Z}[G].$$

  Then $\frac{D(\|\mathbf{r},\mathbf{s},\mathbf{t}\|)}{[R,S,T]}$ is a $\mathbb{Z}/2$-vector space.
- A purely algebraic statement proved using homotopy theory.
- `arXiv:1506.08324`