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Differential expressions with mixed homogeneity and spaces of smooth functions they generate in arbitrary dimension ☆

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A B S T R A C T

Let $\{T_1, \dots, T_J\}$ be a collection of differential operators with constant coefficients on the torus $\mathbb{T}^n$. Consider the Banach space X of functions f on the torus for which all functions $T_j f$, $j = 1, \dots, J$, are continuous. Extending the previous work of the first two authors, we analyze the embeddability of X into some space $C(K)$ as a complemented subspace. We prove the following. Fix some pattern of mixed homogeneity and extract the senior homogeneous parts (relative to the pattern chosen) $\{\sigma_1, \dots, \sigma_J\}$ from the initial operators $\{T_1, \dots, T_J\}$. Let K be the dimension of the linear span of $\{\sigma_1, \dots, \sigma_J\}$. If $K \geqslant 2$, then X is not isomorphic to a complemented subspace of $C(K)$ for any compact space K.

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The main ingredient of the proof of this fact is a new anisotropic embedding theorem of Sobolev type for vector fields.

Contents

0. Introduction

The space $C^{(k)}(\mathbb{T})$ of k times continuously differentiable functions on the unit circle $\mathbb{T}$ of the complex plane is isomorphic to $C(\mathbb{T})$ (modulo constants, an isomorphism is given by k-fold differentiation). However, it has long been known that, already for the 2-dimensional torus $\mathbb{T}^2$, the situation is different. The understanding of this phenomenon has been increasing gradually, starting with [5] and [6], and then through the work done in [7,8,15,23,21,11,9,10,18].

We begin our discussion directly with the general framework considered in [9]. Let $T = \{T_1, \dots, T_J\}$ be a collection of differential operators with constant coefficients on the torus $\mathbb{T}^n$. This means that each T_j is a linear combination of differential monomials $D^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$. Here $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex composed of nonnegative integers. Next, by $\partial_j f$ we mean the operator $g \mapsto \frac{\partial}{\partial t} g(e^{2\pi i t})$, $t \in \mathbb{R}$, applied to f with respect to the jth variable. In general, we use the mapping $t \mapsto e^{2\pi i t}$ to parametrize the unit circle; so, the trigonometric system is $\{e^{2\pi i k t}\}_{k \in \mathbb{Z}}$, $t \in [0, 1)$, and the (multiple) Fourier series are handled accordingly. The quantity $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is called the *order* of the differential monomial. The order of a differential operator is the largest order of a differential monomial involved in the operator.

The above collection T gives rise to the following seminorm on the set of trigonometric polynomials in n variables:

$$\|f\|_T = \max_{1 \leqslant j \leqslant J} \|T_j f\|_{C(\mathbb{T}^n)}.$$

The Banach space determined by this seminorm (i.e., the result of factorization over the null-space and completion) is denoted by $C^T(\mathbb{T}^n)$ and is called the space of smooth functions generated by the collection T.

When T consists of all differential *monomials* of order at most s, we obtain the classical space $C^{(s)}(\mathbb{T}^n)$ of s times continuously differentiable functions on the n-dimensional torus. It is known that if $n \geqslant 2$ and $s \geqslant 1$, then the bidual of this space does not embed in a Banach lattice as a complemented subspace, see [8,15]. In particular, this bidual is not isomorphic to a complemented subspace of $C(K)$ for any compact space K. Note that the space $C^{(s)}(\mathbb{T}^n)$ has the same property *a fortiori* because the bidual of a $C(K)$-space is also a space of type $C(S)$; see, e.g., Section 13.4.3 and formula (1) in p. 481 in [22].

If T is an *arbitrary* finite collection of differential monomials, we deal with the classical *anisotropic* spaces of smooth functions. The isomorphism problem for them was treated in [21,23,11]. Not giving a precise statement, we signalize that the following dichotomy occurs: if the collection T contains a senior differential monomial (i.e., the monomial whose multiindex dominates coordinatewise all other multiindices involved), then, up to some not quite essential subtleties (see [11]; similar subtleties arise in Subsection 4.2 in the present paper), $C^T(\mathbb{T}^n)$ is isomorphic to $C(\mathbb{T}^n)$; otherwise, again, the bidual of $C^T(\mathbb{T}^n)$ is not embeddable complementedly in a $C(K)$-space (more generally, in a Banach lattice).

The importance of the absence of a “senior” operator for nonisomorphism was further emphasized by the results of [9] and [10]. In those papers, the case of a collection T consisting not necessarily of differential monomials was treated for the first time.

The main result of [9] and [10] says the following. Suppose all operators in the collection T are of order not exceeding $s > 0$. In every operator T_j of the collection, we drop its *junior part*, i.e., all differential monomials of order strictly smaller than s. The remaining *senior part* σ_j is a homogeneous differential operator of order precisely s. *If there are two linearly independent operators among the σ_j, $j = 1, \dots, J$, then the bidual of $C^T(\mathbb{T}^n)$ is not isomorphic to a complemented subspace of a $C(K)$-space.*

But if all σ_j are multiples of one of them, the situation was still unclear. More precisely, in the case of the two-dimensional torus, in [9] it was shown that then the space $C^T(\mathbb{T}^2)$ is isomorphic indeed to a $C(K)$-space if the junior parts of all T_j's vanish. (In higher dimension the picture is more complicated.) However, if they do not, nonisomorphism may occur again. We already saw this when we discussed anisotropic spaces of smooth functions: two incomparable maximal monomials (the terms “incomparable” and “maximal” are related to the coordinatewise partial ordering of multiindices) involved in the definition of such a space need not be of one and the same order s, though the space itself is definitely not of type $C(K)$ if two such monomials exist.

This suggests that the concept of mixed homogeneity (permeating the theory of anisotropic spaces) may play a role also in the general situation. The main result of this paper says that it is indeed the case. This can be viewed as a joint refinement of the results of [21,23,11] and [9,10].

Geometrically, a *mixed homogeneity pattern* in n variables is determined by a hyperplane Λ intersecting the n positive coordinate semiaxes. The equation of such a hyperplane is $\sum_{k=1}^{n} \frac{x_k}{a_k} = 1$, where the a_k are positive numbers. A differential operator S is said to be Λ-homogeneous if all points corresponding to the multiindices of differential monomials involved in S belong to Λ. Consider a finite collection $T = \{T_1, \dots, T_J\}$ of differential operators in n variables such that

$$\sum_{k=1}^{n} \frac{\alpha_k}{a_k} \leqslant 1 \tag{0.1}$$

for every differential monomial $\partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$ involved in at least one of T_j. Dropping all terms in T_j for which we have strict inequality in (0.1), we obtain a differential operator σ_j to be called the Λ-senior part of T_j.

Theorem 0.1. *If for at least one choice of Λ there are at least two linearly independent operators in the collection of Λ-senior parts, then the bidual of $C^T(\mathbb{T}^n)$ is not isomorphic to a complemented subspace of a $C(K)$-space.*

It has already been explained that then $C^T(\mathbb{T}^n)$ itself does not embed complementedly in a $C(K)$-space.

In what follows, a hyperplane satisfying (0.1) will be called *admissible* for the collection T. It is easy to realize (and in fact this will be explained in the sequel) that to decide whether or not the theorem is applicable, it suffices to inspect only a certain finite collection of admissible hyperplanes.

It should be noted that, in all the above references except for the early papers [5] and [6], nonisomorphism was proved by combining the techniques of absolutely summing operators (specifically, the Grothendieck theorem and its extensions) with a Sobolev-type embedding theorem. Here we do the same, more precisely, we try to imitate the pattern of [9] and [10]. In our preprint [13], we proved Theorem 0.1 for $n = 2$ in accordance with this plan, and claimed light-heartedly that the general problem could be reduced to that case. Though the phenomenon of nonisomorphism is of two-dimensional nature indeed, the situation has turned out to be not as simple as we thought, and the case of $n > 2$ has required some additional effort. The nature of the difficulty lay in the fact that the proof in the preprint [13] definitely had some “common sense logic”. However, in dimensions greater than 2 that logic led to a deadlock unless revised thoroughly.

In particular, the general case has demanded a more sophisticated embedding theorem than in [13]. That theorem is the second main result of the paper. The statement will be presented in Subsection 1.2. Here, in the Introduction, we only make some comments to convey the flavor of that result. For this, it is convenient to state a consequence of it, which is interesting in itself.

We remind the reader the definition of a Sobolev space with nonintegral smoothness:

$$W_2^{\alpha,\beta}(\mathbb{R}^2) = \{f \in \mathcal{S}'(\mathbb{R}^2)\colon |\xi|^\alpha |\eta|^\beta \hat{f}(\xi,\eta) \in L^2(\mathbb{R}^2)\}. \tag{0.2}$$

Here $\mathcal{S}$ is the Schwartz class and α and β are nonnegative reals. Formally, it is better to define this space as the completion of $\mathcal{S}$ with respect to the norm (0.2), to avoid multiplication of a distribution by a nonsmooth function. Note also that our terminology is slightly nonstandard: quite often, the term "a Sobolev space" is reserved only for the case of derivatives of integral order, and the class we require is called differently.

Theorem 0.2. *Let φ_j, $j = 1, \dots, N$, be compactly supported distributions on the plane $\mathbb{R}^2$. Suppose that they satisfy the system of equations*

$$\begin{cases} -\partial_1^k \varphi_1 = \mu_0; \\ -\partial_1^k \varphi_2 + \partial_2^l \varphi_1 = \mu_1; \\ \vdots = \vdots; \\ -\partial_1^k \varphi_j + \partial_2^l \varphi_{j-1} = \mu_{j-1}; \\ \vdots = \vdots; \\ -\partial_1^k \varphi_N + \partial_2^l \varphi_{N-1} = \mu_{N-1}; \\ \partial_2^l \varphi_N = \mu_N, \end{cases} \tag{0.3}$$

where $\mu_0, \dots, \mu_N$ are finite (compactly supported) measures on the plane. Then

$$\sum_{j=1}^N \|\varphi_j\|_{W_2^{\alpha,\beta}(\mathbb{R}^2)} \lesssim \sum_{j=0}^N \|\mu_j\| \tag{0.4}$$

whenever α and β are nonnegative and satisfy

$$\frac{\alpha + \frac{1}{2}}{k} + \frac{\beta + \frac{1}{2}}{l} = 1. \tag{0.5}$$

As usual, the norm of a measure is its total variation. Here and in what follows, by the symbol "$a \lesssim b$" we mean "$a \leqslant Cb$", where C is some uniform constant.

Theorem 0.2 differs from classical statements by the fact that the condition to be a measure (or an integrable function, which is nearly the same in our setting) is imposed on some linear combinations of derivatives of different functions rather than on certain derivatives themselves of one function. In other words, it is an embedding theorem for vector fields, where differential operators mix different components of a field together.

Such a "mixing" leads to genuine difficulties only for the case of the limiting integrability exponent 1 (i.e., the situation is often much simpler if the relevant differential expressions are assumed to belong to L^p with $p > 1$ instead of being measures or integrable functions). Also, the isotropic and anisotropic cases should be distinguished (in the setting of Theorem 0.2, they correspond to the cases of $k = l$ and $k \neq l$, respectively).

In the isotropic case, there was a great progress in the last decade in understanding the embedding phenomena for vector fields. We quote the pioneering publication [2], the recent paper [26], which is the closest to our setting, and the survey [27]. The full strength of this theory can be felt only in dimension 3 or higher. In dimension 2, the results usually follow via linear-algebraic transformations from the Gagliardo–Nirenberg inequality or, rather, from its bilinear version:

$$\left|\iint\limits_{\mathbb{R}^2} fg\right| \leqslant \iint\limits_{\mathbb{R}^2} |\partial_1 f| \iint\limits_{\mathbb{R}^2} |\partial_2 g|, \tag{0.6}$$

say, for smooth compactly supported functions f and g on $\mathbb{R}^2$. (In fact, we shall deal with a more general case of reasonably regular functions for which the distributional derivatives $\partial_1 f$ and $\partial_1 g$ are measures.) The case of $k = l$ in Theorem 0.2 falls under the scope of the theory mentioned above. However, see [9] for a simple proof (that argument was known to the first author as early as in 1974 and even was included in an unpublished preliminary version of the paper [7]). We also mention the paper [19] in this connection.

In the anisotropic case, embedding theorems are more difficult. We refer the reader to [24] and [14] for anisotropic versions of limit order Sobolev embeddings for individual functions in dimension n. See also [21], were an anisotropic counterpart of the bilinear estimate (0.6) was proved in dimension 2 (consult the paper [25] for further development). However, *we are not aware of any previous work devoted to extension of the anisotropic stuff to vector fields*, like, e.g., in Theorem 0.2. In our preprint [13], we did prove that theorem, but only for the case of $\alpha = \frac{k-1}{2}$, $\beta = \frac{l-1}{2}$. Now we can state it in full generality (homogeneity considerations show that condition (0.5) is also necessary in it), but this is *not the main novelty* in the story: in fact, we can even relax the assumption that the μ_j are measures. It turns our that it suffices to require that the Fourier transforms of these objects be "restrictions" of the Fourier transforms of some measures in a higher dimension to a certain "two-dimensional surface" parametrized by $\mathbb{R}^2$. See Section 2 for the details. Here we only mention two facts. First, this generalization was evoked by the case of dimension greater than 2 in Theorem 0.1 (we simply do not now how to do without it and suspect this is impossible). Second, this generalization applies also to the most important consequence of the classical inequality (0.6), specifically, the inequality $\|f\|_{L^2(\mathbb{R}^2)} \lesssim (\|\partial_1 f\|_{L^1(\mathbb{R}^2)} + \|\partial_2 f\|_{L^1(\mathbb{R}^2)})$: even this case admits a refinement where $\partial_1 f$ and $\partial_2 f$ are not necessarily integrable functions or measures.

It should also be noted that all the above statements concern functions of two variables, but the method is applicable in some other situations. For example, we shall show

how to deduce the inequality

$$\|f\|^2_{L^2(\mathbb{R}^3)} \lesssim \|\partial_1 f\|_{L^1(\mathbb{R}^3)} \|(\partial_2^2 + \partial_3^2) f\|_{L^1(\mathbb{R}^3)},$$

which seems to be new. See Subsection 1.3.

Returning to Theorem 0.1, it is natural to ask what happens if its assumption is violated, that is, the Λ-senior parts of the differential operators are proportional to some of them for an arbitrary choice of an admissible hyperplane Λ. We shall see that, under a certain "ellipticity" condition, the space $C^T(\mathbb{T}^n)$ is isomorphic in this situation to $C(\mathbb{T}^n)$. (We recall that, by the Milyutin theorem, all spaces $C(K)$ for K compact metric and uncountable are mutually isomorphic. So we can always talk about some fixed one, say, $C(\mathbb{T})$ in similar situations.) However, without this ellipticity condition nonisomorphism to a complemented subspace of a $C(K)$-space may occur again, at least for two different reasons. First, a change of variables or (if $n > 2$) the elimination of some variable may sometimes restore the applicability of Theorem 0.1. Second, there is an effect of somewhat different nature, related to the Cohen theorem on idempotents and also leading to nonisomorphism. This effect was first observed in [18] in the case of dimension 3.

However, our analysis will still be not quite complete. The problems remaining are rather of arithmetic nature. See Section 4 for some more details.

The paper is organized as follows. Section 1 is devoted to embedding theorems. It consists of two parts: the first contains a certain "proper" analog of the Gagliardo–Nirenberg inequality (0.6), and in the second we derive (anisotropic) embedding theorems for vector fields. In Section 2 we prove Theorem 0.1. The proof is rather technical, so it is split into several portions. A general outline of the arguments is given in the introductory Subsection 2.1. In Section 3 we store the proofs of some auxiliary statements skipped in Section 2.

We finish the paper with several examples where Theorem 0.1 is not applicable. This stuff is discussed in Section 4, which is divided into two subsections. In Subsection 4.1, we discuss some collections T that are not covered by Theorem 0.1, but for which we can establish nonisomorphism nevertheless. In passing, we introduce some notions, which turn out to be useful in higher dimensions, in particular, in the subsequent Subsection 4.2. There we treat the cases where there is a dominant operator, and it is elliptic in a sense. We state that then the space in question is isomorphic to $C(K)$.

Finally, we address the reader to the monograph [28] for the most part of the Banach space theory stuff mentioned in this paper (the Milyutin theorem, p-absolutely summing operators, the Grothendieck theorem, etc.).

Some results of the present paper were announced in [12].

1. Embedding theorems

The nature of Theorem 0.2 stated in the Introduction suggests that, probably, the passage to Fourier transforms is an obligatory first step in proving it. So, let $\psi_j = \widehat{\varphi_j}$, then system (0.3) takes the form

$$\begin{cases} -(2\pi i\xi_1)^k\psi_1(\xi) & = & \hat{\mu}_0(\xi); \\ -(2\pi i\xi_1)^k\psi_2(\xi) + (2\pi i\xi_2)^l\psi_1(\xi) & = & \hat{\mu}_1(\xi); \\ \vdots & = & \vdots\ ; \\ -(2\pi i\xi_1)^k\psi_j(\xi) + (2\pi i\xi_2)^l\psi_{j-1}(\xi) & = & \hat{\mu}_{j-1}(\xi); \\ \vdots & = & \vdots\ ; \\ -(2\pi i\xi_1)^k\psi_N(\xi) + (2\pi i\xi_2)^l\psi_{N-1}(\xi) & = & \hat{\mu}_{N-1}(\xi); \\ (2\pi i\xi_2)^l\psi_N(\xi) & = & \hat{\mu}_N(\xi). \end{cases} \tag{1.1}$$

We recall that, in our setting, Sobolev spaces are also defined in terms of Fourier transforms. So, in fact, some weighted L^2 norm of the functions ψ_j is estimated *directly* in Theorem 0.2.

We did use a partial case of Theorem 0.2 in the preprint [13] to prove Theorem 0.1 in the dimension 2. However, it turned out that in higher dimensions n a different embedding theorem is required. We state it below. By the way, it will also be applicable for the proof of Theorem 0.1 if $n = 2$. Note that, again, it is two-dimensional in nature and deals with a system like (1.1), but this time an interpretation of the statement in terms of differential operators (by taking inverse Fourier transforms) is at least not obvious. Fortunately, we do not require this.

So, let $\rho_0, \dots \rho_\ell$ be finite (complex) measures on $\mathbb{R}^n$ with $n \geqslant 2$, and let $\theta_2, \dots, \theta_n$ be positive numbers. Consider the following continuous functions on $\mathbb{R}^2$:

$$f_j(\xi_1, \xi_2) = \hat{\rho}_j(\xi_1, |\xi_2|^{\theta_2}, \dots, |\xi_2|^{\theta_n}), \quad j = 0, \dots, \ell. \tag{1.2}$$

Next, let $\eta \geqslant 1$ be another positive constant.

Theorem 1.1. *In the above notation, let functions ψ_j, $j = 1, \dots, \ell$, on the plane satisfy the system of equations*

$$\begin{cases} \xi_1\psi_1(\xi) & = & f_0(\xi); \\ \xi_1\psi_2(\xi) - |\xi_2|^\eta\psi_1(\xi) & = & f_1(\xi); \\ \vdots & = & \vdots\ ; \\ \xi_1\psi_j(\xi) - |\xi_2|^\eta\psi_{j-1}(\xi) & = & f_{j-1}(\xi); \\ \vdots & = & \vdots\ ; \\ \xi_1\psi_\ell(\xi) - |\xi_2|^\eta\psi_{\ell-1}(\xi) & = & f_{\ell-1}(\xi); \\ -|\xi_2|^\eta\psi_\ell(\xi) & = & f_\ell(\xi). \end{cases} \tag{1.3}$$

Then

$$\max_j \left(\int_{\mathbb{R}^2} |\psi_j(\xi)|^2 |\xi_2|^{\eta-1} d\xi \right)^{\frac{1}{2}} \lesssim \max_j \|\rho_j\|.$$

The theorem will be applied only when $\theta_2 = 1$ and the other θ_j's (if there are any, i.e., if $n \geqslant 3$) do not exceed 1. Note that we do need the case where some of them are smaller than 1, so that the "surface" $(\xi_1, \xi_2) \mapsto (\xi_1, |\xi_2|^{\theta_2}, \dots, |\xi_2|^{\theta_n})$ is not "quite smooth". The reality is, however, that not only is the smoothness irrelevant, but nothing beyond measurability is required here (see below).

Both Theorem 0.2 and Theorem 1.1 are proved via estimating certain bilinear forms on the basis of elementary complex analysis. We proceed to the details.

1.1. Estimate for bilinear forms

Our goal in this subsection is to estimate the quantity $\int f\bar{g}H$ (here we deal with an improper integral), where H is a certain weight and f and g are expressed in a certain way via the Fourier transforms of some measures. A bound in terms of the norms of these measures will be ensured, similar to (0.6) to a certain extent, especially if we apply the Plancherel theorem to the left-hand side of (0.6). The precise statement is given in Lemma 1.2 below. The technical preparations that preface the statement are dictated by the applications of Lemma 1.2 to the proofs of embedding theorems for vector fields.

So, let d_1 and d_2 be natural numbers with $d_1, d_2 > 1$. We represent $\mathbb{R}^{d_1}$ as the product $\mathbb{R} \times \mathbb{R}^{d_1-1}$ and denote by $\xi_1 \in \mathbb{R}$ and $\xi^{\{1\}} \in \mathbb{R}^{d_1-1}$ the respective coordinates of a point $\xi \in \mathbb{R}^{d_1}$. Consider a continuous function $h : \mathbb{R}^{d_1-1} \to \mathbb{R}$ and a locally integrable function $H : \mathbb{R}^{d_1-1} \to \mathbb{R}_+$ such that

$$\forall\, a, b \in \mathbb{R}_+ \qquad \int_{\{|h(\zeta)| \in [a,b]\}} H(\zeta)\, d\zeta \lesssim |b-a|. \tag{1.4}$$

Also, let $\Phi : \mathbb{R}^{d_1-1} \to \mathbb{R}^{d_2-1}$ be an arbitrary Borel measurable function.

Next, let ρ_1 and ρ_2 be two finite (complex Borel) measures on $\mathbb{R}^{d_2}$, and let two functions f_1 and f_2 on $\mathbb{R}^{d_1}$ satisfy the equations

$$\begin{aligned} (\xi_1 - \sigma h(\xi^{\{1\}})) f(\xi) &= \hat{\rho}_1(\xi_1, \Phi(\xi^{\{1\}})); \\ (\xi_1 - \tau h(\xi^{\{1\}})) g(\xi) &= \hat{\rho}_2(\xi_1, \Phi(\xi^{\{1\}})), \end{aligned} \tag{1.5}$$

where σ and τ are complex numbers with nonzero imaginary parts of different signs. Finally, for $\varepsilon, R > 0$ we introduce the domain

$$\Omega_{\varepsilon,R} = \{\xi \in \mathbb{R}^{d_1} : |h(\xi^{\{1\}})| \geqslant \varepsilon, |\xi^{\{1\}}| \leqslant R\}. \tag{1.6}$$

Lemma 1.2. *Under the above assumptions, we have*

$$\Big| \int\limits_{\Omega_{\varepsilon,R}} f(\xi)\overline{g(\xi)} H(\xi^{\{1\}})\, d\xi \Big| \leqslant C\|\rho_1\|\|\rho_2\|,$$

where the constant C depends only on σ, τ, and the (implicit) constant in (1.4). In particular, the estimate is uniform in ε and R.

Surely, by the norm of a measure we mean its total variation. We refer the reader to [25] for a more thorough account of certain bilinear estimates similar to the above.

Proof. Acting formally, we use (1.5) to rewrite the integral in question as follows:

$$I_{\varepsilon,R} \stackrel{\text{def}}{=} \int\limits_{\Omega_{\varepsilon,R}} \frac{\hat{\rho}_1\big(\xi_1, \Phi(\xi^{\{1\}})\big)\overline{\hat{\rho}_2\big(\xi_1, \Phi(\xi^{\{1\}})\big)} H(\xi^{\{1\}})\, d\xi}{\big(\xi_1 - \sigma h(\xi^{\{1\}})\big)\big(\xi_1 - \overline{\tau} h(\xi^{\{1\}})\big)}.$$

Then we use the definition of $\hat{\rho}_1$ and $\hat{\rho}_2$ and (still formally) change the order of integration to obtain

$$I_{\varepsilon,R} = \int\limits_{\mathbb{R}^{d_2}\times\mathbb{R}^{d_2}} K_{\varepsilon,R}(x,y)\rho_1(dx)\overline{\rho_2(dy)}, \tag{1.7}$$

where $K_{\varepsilon,R}$ is calculated by the formula

$$K_{\varepsilon,R}(x,y) = \int\limits_{\Omega_{\varepsilon,R}} \frac{e^{2\pi i\left((x_1-y_1)\xi_1 + \langle x^{\{1\}} - y^{\{1\}}, \Phi(\xi^{\{1\}})\rangle\right)} H(\xi^{\{1\}})\, d\xi}{\big(\xi_1 - \sigma h(\xi^{\{1\}})\big)\big(\xi_1 - \overline{\tau} h(\xi^{\{1\}})\big)} \tag{1.8}$$

and the angular brackets denote the inner product in $\mathbb{R}^{d_2-1}$. All these operations are justified easily if done in the reverse order: plug (1.8) in (1.7), discover that the resulting multiple integral converges absolutely for every fixed ε and R because of the assumptions about σ and τ, and apply the Tonelli and Fubini theorems. Surely, this does not yield a uniform upper bound for all ε and R yet.

Now, to obtain such an upper bound, it suffices to show that the functions (1.8) are bounded uniformly in all ε, R, x, and y. Fixing $\xi^{\{1\}}$, we calculate the integral with respect to ξ_1 separately. For this, we consider the following meromorphic function of one complex variable:

$$z \mapsto \frac{e^{2\pi i(x_1-y_1)z}}{\big(z - \sigma h(\xi^{\{1\}})\big)\big(z - \overline{\tau} h(\xi^{\{1\}})\big)}.$$

We shall assume that $x_1 - y_1 \geqslant 0$ (in the opposite case, we argue similarly, replacing the upper half-plane with the lower one). So, in the half-plane $\operatorname{Im} z > 0$, this function is

$O(|z|^{-2})$. By the assumption imposed on the imaginary parts of σ and τ, it has either two poles in the upper half-plane (at $\sigma h(\xi^{\{1\}})$ and $\overline{\tau} h(\xi^{\{1\}})$) or none. We represent the integral $\int_{-\infty}^{\infty}$ as $\lim_{r\to\infty}\int_{-r}^{r}$, apply the Cauchy residue theorem to the contour composed of the segment $[-r, r]$ and the upper half-circle of radius r and centered at zero, and pass to the limit as $r \to \infty$, obtaining

$$\int_{\mathbb{R}} \frac{e^{2\pi i(x_1-y_1)\xi_1}\,d\xi_1}{\big(\xi_1 - \sigma h(\xi^{\{1\}})\big)\big(\xi_1 - \overline{\tau} h(\xi^{\{1\}})\big)} = 2\pi i \frac{e^{2\pi i(x_1-y_1)\sigma h(\xi^{\{1\}})} - e^{2\pi i(x_1-y_1)\overline{\tau} h(\xi^{\{1\}})}}{(\sigma - \overline{\tau}) h(\xi^{\{1\}})}$$

if $\operatorname{Im}\sigma h(\xi^{\{1\}}), \operatorname{Im}\tau h(\xi^{\{1\}}) > 0$ (the set of such $\xi^{\{1\}}$ will be denoted by A_+) and

$$\int_{\mathbb{R}} \frac{e^{2\pi i(x_1-y_1)\xi_1}\,d\xi_1}{\big(\xi_1 - \sigma h(\xi^{\{1\}})\big)\big(\xi_1 - \overline{\tau} h(\xi^{\{1\}})\big)} = 0$$

otherwise. That is,

$$K_{\varepsilon,R}(x,y) = 2\pi i \times \int\limits_{\substack{A_+\cap\{|h(\xi^{\{1\}})|\geqslant\varepsilon,\\ |\xi^{\{1\}}|\leqslant R\}}} \frac{\Big(e^{2\pi i(x_1-y_1)\sigma h(\xi^{\{1\}})} - e^{2\pi i(x_1-y_1)\overline{\tau} h(\xi^{\{1\}})}\Big) e^{2\pi i\langle x^{\{1\}}-y^{\{1\}},\Phi(\xi^{\{1\}})\rangle} H(\xi^{\{1\}})\,d\xi^{\{1\}}}{(\sigma-\overline{\tau}) h(\xi^{\{1\}})}.$$

Observe that this expression is equal to zero if $x_1 = y_1$, so in what follows we assume that $x_1 - y_1 > 0$. We split A_+ into a disjoint union of sets as follows:

$$A_+ = \cup_{n\in\mathbb{N}} A_{n,x,y}, \quad A_{n,x,y} = \left\{\xi^{\{1\}} \in A_+ :\ \big|h(\xi^{\{1\}})\big| \in \Big[\frac{n-1}{x_1-y_1}, \frac{n}{x_1-y_1}\Big)\right\},$$

and rewrite the integral accordingly:

$$\int\limits_{A_+\cap\{|h(\xi^{\{1\}})|\geqslant\varepsilon,|\xi^{\{1\}}|\leqslant R\}} \dots = \sum_{n=1}^{\infty} \int\limits_{A_{n,x,y}\cap\{|h(\xi^{\{1\}})|\geqslant\varepsilon,|\xi^{\{1\}}|\leqslant R\}} \dots. \tag{1.9}$$

The integrand can easily be estimated:

$$\left|\frac{\Big(e^{2\pi i(x_1-y_1)\sigma h(\xi^{\{1\}})} - e^{2\pi i(x_1-y_1)\overline{\tau} h(\xi^{\{1\}})}\Big) e^{2\pi i\langle x^{\{1\}}-y^{\{1\}},\Phi(\xi^{\{1\}})\rangle} H(\xi^{\{1\}})}{(\sigma-\overline{\tau}) h(\xi^{\{1\}})}\right| \lesssim$$

$$\begin{cases} |x_1-y_1| H(\xi^{\{1\}}), & \xi^{\{1\}} \in A_{1,x,y}; \\ |x_1-y_1| \frac{(e^{-|\operatorname{Im}\sigma| n} + e^{-|\operatorname{Im}\tau| n}) H(\xi^{\{1\}})}{n}, & \xi^{\{1\}} \in A_{n,x,y}. \end{cases} \tag{1.10}$$

We have used the inequality $|e^a - e^b| \lesssim |a-b|$, $|a|, |b| \lesssim 1$ in the first case and the inequality $|e^a - e^b| \lesssim e^{\operatorname{Re} a} + e^{\operatorname{Re} b}$ in the second (the oscillating exponential involving Φ was replaced by 1 in the two cases). Thus,

$$|K_\varepsilon(x,y)| \overset{(1.9)}{\underset{(1.10)}{\lesssim}} |x_1 - y_1| \int\limits_{A_{1,x,y} \cap \{|h(\xi^{\{1\}})| \geqslant \varepsilon,\, |\xi^{\{1\}}| \leqslant R\}} H(\xi^{\{1\}})\, d\xi^{\{1\}} + \sum_{n=2}^{\infty} |x_1 - y_1| \frac{(e^{-|\operatorname{Im}\sigma| n} + e^{-|\operatorname{Im}\tau| n})}{n} \int\limits_{A_{n,x,y}} H(\xi^{\{1\}})\, d\xi^{\{1\}} \overset{(1.4)}{\lesssim} 1 + \sum_{n=2}^{\infty} \frac{e^{-|\operatorname{Im}\sigma| n} + e^{-|\operatorname{Im}\tau| n}}{n} \lesssim 1.$$

This proves the lemma. □

Remark 1.3. It should be noted that Lemma 1.2 remains true if we allow σ and τ to depend on $\xi^{\{1\}}$ in a not too complicated way, still assuming that their imaginary parts are nonzero and have different signs everywhere. We shall only use this observation in the case where σ and τ take finitely many values and obey the above assumption (and, surely, are Borel measurable). Then it suffices to split the domain of integration into maximal constancy sets for the pair (σ, τ) and to do the same as before on each of these sets separately.

1.2. Embedding theorems for vector fields

We pass to an application of Lemma 1.2 which will lead to Theorems 0.2 and 1.1.

Theorem 1.4. *Let $N > 1$ be a positive integer. Suppose that the numbers d_1, d_2 and the functions h, H, Φ are the same as in Lemma 1.2. Let $\rho_0, \rho_1, \ldots, \rho_N$ be finite (complex) Borel measures on $\mathbb{R}^{d_2}$ and let functions $g_0, \ldots, g_N$ on $\mathbb{R}^{d_1}$ be given by*

$$g_j(\xi) = \hat{\rho}_j(\xi_1, \Phi(\xi^{\{1\}})), \quad j = 0, 1, \ldots, N.$$

Assume that functions $f_1, f_2, \ldots, f_N$ defined on $\mathbb{R}^{d_1}$ satisfy the system of equations

$$\begin{cases} \xi_1 f_1(\xi) &= \quad g_0(\xi); \\ \xi_1 f_2(\xi) - h(\xi^{\{1\}}) f_1(\xi) &= \quad g_1(\xi); \\ \quad\vdots &= \quad \vdots; \\ \xi_1 f_j(\xi) - h(\xi^{\{1\}}) f_{j-1}(\xi) &= g_{j-1}(\xi); \\ \quad\vdots &= \quad \vdots; \\ \xi_1 f_N(\xi) - h(\xi^{\{1\}}) f_{N-1}(\xi) &= g_{N-1}(\xi); \\ \qquad - h(\xi^{\{1\}}) f_N(\xi) &= \quad g_N(\xi). \end{cases} \tag{1.11}$$

Then

$$\sum_{j=1}^{N} \Big(\int\limits_{\mathbb{R}^{d_1}} |f_j|^2(\xi) H(\xi^{\{1\}})\, d\xi \Big)^{\frac{1}{2}} \lesssim \sum_{j=0}^{N} \|\rho_j\|.$$

Proof. We shall deduce this statement from Lemma 1.2 with the help of elementary algebraic transformations. Let σ be a complex number. For every j, we multiply the line involving g_j in the system (1.11) by σ^j and add the resulting formulas. This will lead to the equation

$$\big(\xi_1 - \sigma h(\xi^{\{1\}})\big) f_\sigma = g_\sigma, \quad f_\sigma = \sum_{j=1}^{N} \sigma^{j-1} f_j, \quad g_\sigma = \sum_{j=0}^{N} \sigma^j g_j. \tag{1.12}$$

Writing the same equation for another complex number τ,

$$\big(\xi_1 - \tau h(\xi^{\{1\}})\big) f_\tau = g_\tau, \quad f_\tau = \sum_{j=1}^{N} \tau^{j-1} f_j, \quad g_\tau = \sum_{j=0}^{N} \tau^j g_j, \tag{1.13}$$

we arrive at equations (1.5) of Lemma 1.2. Indeed, it suffices to observe that $g_\sigma(\xi) = \hat{\rho}_\sigma(\xi_1, \Phi(\xi^{\{1\}}))$ and $g_\tau(\xi) = \hat{\rho}_\tau(\xi_1, \Phi(\xi^{\{1\}}))$, where the measures ρ_σ and ρ_τ are obtained from the measures ρ_j in the same way as the functions g_σ and g_τ are obtained from the g_j. Now, assume that σ and τ have nonzero imaginary parts of different signs. By Lemma 1.2, we have

$$\Big| \int\limits_{\Omega_{\varepsilon,R}} f_\sigma(\xi) \overline{f_\tau(\xi)} H(\xi^{\{1\}})\, d\xi \Big| \lesssim \|\rho_\sigma\| \|\rho_\tau\| \lesssim \Big(\sum_{j=0}^{N} \|\rho_j\| \Big)^2. \tag{1.14}$$

Next, take N pairwise different complex numbers σ_s, $s = 1, 2, \dots, N$, in the upper half-plane and N pairwise different complex numbers τ_t, $t = 1, 2, \dots, N$, in the lower half-plane. Each f_j is a linear combination of the functions f_{σ_s} (to see this, it suffices to resolve a system of linear equations with a Vandermonde determinant). For the same reason, each f_j is a linear combination of the functions f_{τ_t}. Hence, (1.14) yields

$$\int\limits_{\Omega_{\varepsilon,R}} f_j(\xi)\overline{f_j(\xi)}H(\xi^{\{1\}})\,d\xi \lesssim \Big(\sum_{j=0}^{N}\|\rho_j\|\Big)^2$$

for all j, uniformly in ε and R. Now, we observe that, by (1.4), the integral of H over the set $\{\zeta \in \mathbb{R}^{d_1-1}\colon h(\zeta)=0\}$ is zero. Consequently, the integral on the left in the preceding display tends to

$$\int\limits_{\mathbb{R}^{d_1}} |f_j(\xi)|^2 H(\xi^{\{1\}})\,d\xi$$

as $\varepsilon \to 0$ and $R \to \infty$, and we are done. $\square$

Theorem 1.1 is an immediate consequence of the above statement: it suffices to put $d_1 = 2$ (so, naturally, we write $\xi = (\xi_1, \xi_2)$ in place of $\xi = (\xi_1, \xi^{\{1\}})$), $d_2 = n-1$, $\Phi(\xi_2) = (|\xi_2|^{\theta_2}, \dots, |\xi_2|^{\theta_n})$, $h(\xi_2) = |\xi_2|^{\gamma}$, and $H(\xi_2) = |\xi_2|^{\gamma-1}$.

Contrary to the above, the proof of Theorem 0.2 requires the identity mapping of $\mathbb{R}$ into itself in the role of Φ, but otherwise is slightly more tricky. First, we need a refinement of Lemma 1.2.

Remark 1.5. In Lemma 1.2, the function h can be complex-valued if its modulus is continuous and the argument takes finitely many values. Indeed, then the left-hand sides of (1.12) and (1.13) become $\big(\xi_1 - \sigma\exp(i\delta(\xi^{\{1\}}))|h(\xi^{\{1\}})|\big)f_\sigma$ and $\big(\xi_1 - \tau\exp(i\delta(\xi^{\{1\}}))|h(\xi^{\{1\}})|\big)f_\sigma$, where δ is a real function taking finitely many values. Now, if we choose σ and τ so as to their arguments differ by π and the functions $\sigma\exp(i\delta(\xi^{\{1\}}))$ and $\tau\exp(i\delta(\xi^{\{1\}}))$ be nowhere real (which is clearly possible in a plenty of ways), we discover that Remark 1.3 is applicable, so that we again obtain (1.14), etc.

Proof of Theorem 0.2. As has already been explained, we take Fourier transforms and deal with system (1.1). It suffices to prove the theorem for the border cases of α and β, i.e., the cases where $(\alpha,\beta) = (0, l - \frac{l}{2k} - \frac{1}{2})$ and $(\alpha,\beta) = (k - \frac{k}{2l} - \frac{1}{2}, 0)$. They are symmetric, so we analyze only the first one. For each $j = 1,2,\dots,N$, we introduce k functions ψ_j^p, $p = 0,1,2,\dots,k-1$ such that $\psi_j^p(\xi_1,\xi_2) = (i\xi_1)^{k-p-1}|\xi_2|^{\frac{lp}{k}}(c(\operatorname{sgn}\xi_2))^p\psi_j$, where $c(\operatorname{sgn}\xi_2)$ is the kth root of $i\operatorname{sgn}\xi_2$ with the smallest argument. It is easy to see that these new functions satisfy the equations

$$-(i\xi_1)\psi_j^p + |\xi_2|^{\frac{l}{k}}c(\operatorname{sgn}\xi_2)\psi_j^{p-1} = 0, \quad p = 1,2,\dots,k-1;$$
$$-(i\xi_1)\psi_j^0 = -(i\xi_1)^k\psi_j; \quad |\xi_2|^{\frac{l}{k}}c(\operatorname{sgn}\xi_2)\psi_j^{k-1} = (i\xi_2)^l\psi_j.$$

Together with equations (1.1), these equations constitute a system of type (1.11) for the kN functions ψ_j^p with $h(\xi_2) = -ic(\operatorname{sgn}\xi_2)|\xi_2|^{\frac{l}{k}}$, $\Phi(\xi_2) = \xi_2$. On the right-hand side of this system, there are many zeros and the functions $\mathcal{F}[\mu_j]$. Taking $H(\xi_2) = |\xi_2|^{\frac{l}{k}-1}$ and using Theorem 1.4 and Remark 1.5, we get the estimate

$$\sum_{p,j} \|\psi_j^p\|_{L^2(|\xi_2|^{\frac{l}{k}-1})} \lesssim \sum_{j=0}^{N} \|\mu_j\|.$$

Let $j = k-1$, then

$$\|\psi_j^{k-1}\|_{L^2(|\xi_2|^{\frac{l}{k}-1})} = \iint\limits_{\mathbb{R}^2} \Big|\psi_j(\xi)|\xi_2|^{\frac{l(k-1)}{k}}\Big|^2 |\xi_2|^{\frac{l}{k}-1}\, d\xi = \|\psi_j\|_{W_2^{(0,l+\frac{l}{2k}-\frac{1}{2})}}. \qquad \square$$

Remark 1.6. A similar version of Theorem 0.2 holds for the torus $\mathbb{T}^2$. It can be derived from the statement for the plane. However, the corresponding "transference principle" is not immediate: see [13], Subsection 3.2 for the details, where a similar procedure was performed for the case of $(\alpha, \beta) = (\frac{k-1}{2}, \frac{l-1}{2})$. We note that these arguments do not work for the case of bilinear theorems from Subsection 1.1, we do not know whether similar statements hold true for the torus.

1.3. Digression

Two embedding theorems proved above are about functions of two variables. However, the method allows us to obtain certain statements in higher dimensions. We present a fairly simple one as a sample, though it will not be required in the sequel. In it, Φ will be the identity mapping and h will be a polynomial. These features enable us to pass to inverse Fourier transforms and to state the result in terms of derivatives. More general statements in the same spirit are available, but we do not dwell on them here.

Specifically, let $d_1 = d_2 = 3$, and let $h(\xi_2, \xi_3) = \xi_2^2 + \xi_3^2$. Then H can be taken identically equal to 1. Indeed, condition (1.4) can be verified by passing to polar coordinates (we assume that $b > a$):

$$\int\limits_{\{\xi_2^2+\xi_3^2 \in [a,b]\}} d\xi_2\, d\xi_3 = \int\limits_{\sqrt{a}}^{\sqrt{b}} \int\limits_0^{2\pi} r\, d\varphi\, dr = \pi(b-a) \lesssim |b-a|.$$

As has already been said, we take the identity mapping for Φ. Then, by Lemma 1.2 and the Plancherel theorem, for every Schwartz class functions f and g we obtain

$$\Big|\langle f, g\rangle_{L_2(\mathbb{R}^3)}\Big| = \Big|\langle \hat{f}, \hat{g}\rangle_{L_2(\mathbb{R}^3)}\Big| \leqslant C_{\sigma,\tau} \Big\|\big(i\partial_1 - \sigma(\partial_2^2 + \partial_3^2)\big) f\Big\|_{L^1(\mathbb{R}^3)} \times \Big\|\big(i\partial_1 - \tau(\partial_2^2 + \partial_3^2)\big) g\Big\|_{L^1(\mathbb{R}^3)}, \quad \operatorname{Im}\sigma \operatorname{Im}\tau < 0.$$

Putting here $f = g$ and then estimating the right-hand side in an elementary way, we arrive at

$$\|f\|_{L^2(\mathbb{R}^3)} \lesssim \|\partial_1 f\|_{L^1(\mathbb{R}^3)} + \|(\partial_2^2 + \partial_3^2)f\|_{L^1(\mathbb{R}^3)}.$$

It is not difficult to see that this inequality can be rewritten in the form

$$\|f\|^2_{L^2(\mathbb{R}^3)} \lesssim \|\partial_1 f\|_{L^1(\mathbb{R}^3)} \|(\partial_2^2 + \partial_3^2)f\|_{L^1(\mathbb{R}^3)}.$$

Indeed, the latter inequality implies the former immediately. As to the reverse implication, it suffices to replace $f(x)$ by $f(tx)$ in the former inequality and, after the change of variables $y = tx$ in all integrals, put $t = \|\partial_1 f\|_{L^1}/\|(\partial_1^2 + \partial_2^2)f\|_{L^1}$. We do not know whether these inequalities occurred in the literature before. We only mention that, apparently, they are not deducible from the fine and sharp results of the classical papers [14,24].

2. Nonisomorphism

In what follows, we agree not to distinguish notationally between an algebraic polynomial in n variables and the corresponding differential polynomial obtained by replacing the variables with the differentiations $\{\partial_j\}_{j=1,\dots,n} = D$. In particular, we shall sometimes talk about the space $C^T(\mathbb{T}^n)$ corresponding to a collection T of algebraic polynomials. However, any differential polynomial $P(D)$ gives rise to yet another polynomial $\tilde{P}$ called its *characteristic polynomial* and determined by the relation $\widehat{P(D)f} = \tilde{P}\hat{f}$. Thus, under our normalization for the Fourier transformation, $\tilde{P}(\xi) = P(2\pi i\xi_1, \dots, 2\pi i\xi_n)$.

2.1. The plot

A distribution F on the torus $\mathbb{T}^n$ is said to be *proper* if $\hat{F}(k_1, \dots, k_n) = 0$ whenever at least one of the indices k_j is equal to 0. Accordingly, we shall often talk about proper measures or proper integrable functions. With each distribution G, we associate its *proper part* $\sum_{k_1,\dots,k_n \neq 0} \hat{G}(k_1, \dots, k_n) z_1^{k_1} \dots z_n^{k_n}$. The operation of taking the proper part can be expressed in terms of a linear combination of convolutions with certain measures on $\mathbb{T}^n$. So, it is a continuous projection in very many natural spaces on $\mathbb{T}^n$, in particular, in $C^{\{T_1,\dots,T_J\}}(\mathbb{T}^n)$. Thus, to prove that the second dual of $C^{\{T_1,\dots,T_J\}}(\mathbb{T}^n)$ does not embed complementedly in a $C(K)$-space, it suffices to prove the same for the second dual of the subspace $C_0^{\{T_1,\dots,T_J\}}(\mathbb{T}^n)$ consisting of proper functions. In what follows, we shall mainly work with this subspace, because this is more convenient technically.

Consider the natural embedding $f \mapsto \{T_1 f, \dots, T_J f\}$ of the space $C_0^{\{T_1,\dots,T_J\}}(\mathbb{T}^n)$ into the direct sum of J copies of the space $C(\mathbb{T}^n)$, and let $\mathcal{X}$ be the image of $C_0^{\{T_1,\dots,T_J\}}(\mathbb{T}^n)$ under this embedding. By a Banach space theory commonplace (see Subsection 3.5 for more explanations), if the bidual of $C_0^{\{T_1,\dots,T_J\}}(\mathbb{T}^n)$ is isomorphic to a complemented subspace of a $C(K)$-space, then the annihilator of $\mathcal{X}$ in $C(\mathbb{T}^n) \oplus \dots \oplus C(\mathbb{T}^n)$ is complemented in the dual space, which consists of J-tuples of measures on $\mathbb{T}^2$. By the Grothendieck theorem, then an arbitrary bounded linear operator from $\mathcal{X}^\perp$ to a Hilbert space would have been 1-absolutely summing, and, *a fortiori*, 2-absolutely summing (the definition will be recalled shortly). We shall show that this is not the case.

The bounded linear operator which is going to fail to be 2-absolutely summing will be constructed in the following way. With a collection $(\mu_1, \dots, \mu_J) \in \mathcal{X}^\perp$ of measures on the torus, we shall associate a certain collection $\{\rho_s\}$ of finite measures on the plane. Also, we choose *two* variables in a special way among the n available. After renumbering, we may assume that those with the indices 1 and 2 are chosen, and we introduce the functions (1.2). The condition $(\mu_1, \dots, \mu_J) \in \mathcal{X}^\perp$ will ensure eventually that system (1.3) is uniquely solvable, and, by Theorem 1.1, its solution lies in a certain weighted L^2-space. The operator we are looking for takes the initial collection $(\mu_1, \dots, \mu_J)$ to this solution. Precisely this choice of *two* "distinguished" variables reflects the feature (already mentioned several times) that the nature of the result is two-dimensional.

Our next goal will be to show that this operator is not 2-absolutely summing indeed. We remind the reader (see, e.g., [28] or [16] for more details) that an operator $S\colon A \to B$ is said to be 2-absolutely summing if it takes weakly 2-summable sequences to 2-summable sequences. A sequence is said to be 2-summable if the squares of the norms of its elements form an absolutely convergent series. A sequence $\{x_j\}_{j\in\mathbb{N}}$ in A is said to be weakly 2-summable if the series $\sum_j |F(x_j)|^2$ converges for an arbitrary bounded linear functional F on A. So, we must exhibit a weakly 2-summable sequence in $\mathcal{X}^\perp$ that is taken by the operator in question to a sequence that is not 2-summable. There is a simple way to find weakly 2-summable sequences in spaces of measures. Specifically, it is clear that a bounded orthonormal sequence in a Hilbert space is weakly 2-summable. Now, suppose we are given some measures σ_k on a space K that are all absolutely continuous with respect to one finite measure μ on K. If their densities form a bounded orthogonal system in $L^2(\mu)$, then the σ_k form a weakly 2-summable sequence in $M(K)$. Indeed, the mapping $f \mapsto f d\mu$ is continuous from $L^2(\mu)$ to $M(K)$, and the property of being a weakly 2-summable sequence survives under the action of a bounded linear operator. Using this observation, we shall construct a special sequence of elements of $\mathcal{X}^\perp$ merely from the characters of the torus; it will be weakly 2-summable by orthogonality.

It will turn out that the operator mentioned above takes this sequence to one that is not 2-summable, as required. Surely, the condition that the senior Λ-homogeneous parts of the differential polynomials in the collection T span a linear space of dimension at least 2 will play a crucial role in all that. However, the use of this condition is related to some technicalities, see the details below. We only mention that we shall work with a modification of a collection T rather than with this collection itself. The trivial observation that T can be replaced with any other collection with the same linear span will be used several times, but, besides of this, we shall also pass to certain complemented subspaces of $C_0^{\{T_1,\dots,T_J\}}(\mathbb{T}^n)$.

As has already been said, Theorem 1.1 will play a crucial role in the construction of the "bad" operator from the annihilator to a Hilbert space. The parameters γ and θ_j

occurring in that theorem will be defined in terms of the admissible hyperplane in question. However, before defining these parameters, we also need to modify that hyperplane. We shall start with the detailed exposition of that.

Mainly, we work on the torus $\mathbb{T}^n$ with $n \geqslant 3$, but the arguments are also applicable in the case of two variables with simplifications. Sometimes the simplifications are indicated explicitly, sometimes they are left to the reader. The resulting proof for $n = 2$ differs somewhat from the proof for $n = 2$ exposed in the preprint [13].

To lighten the presentation below, we postpone the proofs of some auxiliary facts till Section 3. The reader who prefers a linearly ordered exposition may address to a due subsection of Section 3 immediately after the particular statement.

2.2. Hyperplane

Let $\mathcal{M}$ denote the set of multiindices corresponding to the monomials occurring in some operator of the collection T. Suppose we have an admissible hyperplane that passes through several points of $\mathcal{M}$. We claim that this hyperplane can be rotated slightly in such a way that it remains admissible, its intersection with $\mathcal{M}$ does not change, but it becomes rational. The precise statement looks like this.

Lemma 2.1. *Let $X_0, X_1, \dots, X_N$ and $Z_0, Z_1, \dots, Z_M$ be two collections of vectors in $\mathbb{R}^n$ whose coordinates are nonnegative integers. Let p be a vector with positive coordinates such that*

$$\langle X_j, p\rangle = 1, \quad j = 0, 1, \dots, N; \quad \langle Z_j, p\rangle < 1, \quad j = 0, 1, \dots, M. \tag{2.1}$$

Then there exists a rational vector p_1 with positive coordinates such that formulas (2.1) remain true with p changed for p_1 and the affine hyperplane $\tilde{L} = \{X\colon \langle X, p_1\rangle = 1\}$ contains at least n linearly independent integral points.

This statement is very easy to believe. We postpone the routine accurate proof till Subsection 3.1. Now we want to describe some additional steps.

After the modification done with the help of the lemma, formulas (2.1) can be rewritten as

$$\langle X_j, P\rangle = a, \quad j = 0, 1, \dots, N; \quad \langle Z_j, P\rangle < a, \quad j = 0, 1, \dots, M,$$

where a is a natural number and $P = (P_1, \dots, P_n)$ is a vector with natural coordinates. We shift the entire configuration by a vector Y. The resulting points satisfy the conditions

$$\langle X_j + Y, P\rangle = a + \langle Y, P\rangle, \quad j = 0, 1, \dots, N;$$
$$\langle Z_j + Y, P\rangle < a + \langle Y, P\rangle, \quad j = 0, 1, \dots, M.$$

The shifted hyperplane intersects the kth coordinate axis at the point whose coordinate is $\frac{a+\langle Y,P\rangle}{P_k}$. Let C be some large positive integer. We choose the vector Y in the following way:

$$Y = (CP_1P_2 \dots P_n, CP_1P_2 \dots P_n, \dots, CP_1P_2 \dots P_n) - X_0.$$

This vector has positive coordinates (if C is sufficiently large), and the numbers $\frac{a+\langle Y,P\rangle}{P_k}$ are positive integers for this choice of Y. Therefore, we can shift the hyperplane (together with $\mathcal{M}$) in such a fashion that it intersects the coordinate axes at points with natural coordinates.

This shift of $\mathcal{M}$ corresponds to multiplication of all polynomials in the collection T by a monomial. Shortly, it will be explained that this operation is innocent.

2.3. Preservation of linear independence after reduction of the number of variables

We recall some notation. The collection $T = \{T_j\}_{j=1,2,\dots,J}$ generating the space $C^T(\mathbb{T}^n)$ consists of the operators

$$T_j = \sum_{\langle a,\alpha\rangle \leqslant k} c_{\alpha,j} D^\alpha. \tag{2.2}$$

The vector a and the number k occurring in this formula determine the admissible hyperplane in question. The coordinates of a will be denoted by $a_1, a_2, \dots, a_n$. The senior part σ_j of T_j is formed by the terms with $\langle a, \alpha\rangle = k$ in (2.2). The sum τ_j of all other terms is the junior part of T_j. We are going to prove the following statement.

Theorem 2.2. *If there are at least two linearly independent operators among the σ_j, $j = 1, \dots, J$, then $C^T(\mathbb{T}^n)^{**}$ does not embed complementedly in a $C(K)$-space.*

In this subsection, we describe certain modifications of the collection T.

It has already been mentioned that it suffices to prove the same claim for the subspace $C_0^T(\mathbb{T}^n)$ of $C^T(\mathbb{T}^n)$ that consists of proper functions, i.e., of the functions f with $\widehat{f}(b) = 0$ whenever at least one of the coordinates of the multiindex b vanishes). Let $s_1, \dots, s_n \in \mathbb{N}$, and consider the shifted collection $S = \{\partial_1^{s_1} \dots \partial_n^{s_n} T_j \colon j = 1, \dots, J\}$. The setting of proper functions has an advantage that, clearly, the spaces $C_0^T(\mathbb{T}^n)$ and $C_0^S(\mathbb{T}^n)$ are isomorphic (specifically, the mapping $f \mapsto \partial_1^{s_1} \dots \partial_n^{s_n} f$ is an isomorphism of the first space onto the second, which may fail for the spaces without the subscript "0"). The passage to the second space corresponds to the translation by $(s_1, \dots, s_n)$ of the multiindices of all differential monomials involved in the T_j, $j = 1, \dots, J$. Addressing to Lemma 2.1 and the arguments after it, we see that the coordinates of the vector a in (2.2) can be made natural numbers, the same can be done with k, and after that it can be arranged that the affine hyperplane $\{u \colon \langle a, u\rangle = k\}$ intersect all positive coordinate semiaxes at integral points. All this will ensure as a byproduct that $a_j > 0$ for all j.

Renumbering the coordinates, we may assume that $a_1 \geqslant a_2 \geqslant a_3 \geqslant \ldots \geqslant a_n > 0$. The first two variables in this enumeration will play a special role in the proof.

In what follows, we shall assume that all that is arranged from the outset, and we shall prove the theorem for $X = C_0^T(\mathbb{T}^n)$ in place of $C^T(\mathbb{T}^n)$. To do this, we need a lemma. In the sequel, we work rather with the adjoints $T_j^* = \sigma_j^* + \tau_j^*$ than the operators T_j themselves; the characteristic polynomials of the adjoints look like this:

$$P_j(\xi_1, \ldots, \xi_n) = \sum_{\langle a,\alpha\rangle \leqslant k} c_{\alpha,j}(2\pi i\xi_1)^{\alpha_1} \ldots (2\pi i\xi_n)^{\alpha_n}(-1)^{|\alpha|} =$$
$$= \sum_{\langle a,\alpha\rangle = k} \ldots + \sum_{\langle a,\alpha\rangle < k} \ldots \stackrel{\text{def}}{=} \Pi_j(\xi_1, \ldots, \xi_n) + \rho_j(\xi_1, \ldots, \xi_n). \quad (2.3)$$

(Thus, Π_j is the characteristic polynomial of σ_j^* and ρ_j is that of τ_j^*.) Clearly, the dimension of the linear span of the Π_j is at least 2.

Lemma 2.3. *There is no loss of generality in assuming that, together with the conditions listed above, the following one is fulfilled: among the operators $\Pi_j(\xi_1, \ldots, \xi_n)$, the first two have the property that the polynomials*

$$\Pi_1(\xi_1^{a_1}, \xi_2^{a_2}, \ldots, \xi_2^{a_n}) \text{ and } \Pi_2(\xi_1^{a_1}, \xi_2^{a_2}, \ldots, \xi_2^{a_n})$$

are linearly independent.

Remark 2.4. It is quite easy to realize that the polynomials $\Pi_1(\xi_1, \xi_2^{a_2}, \ldots, \xi_2^{a_n})$ and $\Pi_2(\xi_1, \xi_2^{a_2}, \ldots, \xi_2^{a_n})$ are also linearly independent, provided the claim of the lemma holds true; this will be used in what follows.

If the number of variables is 2, the supplementary condition mentioned in the lemma looks like this: $\Pi_1(\xi_1^{a_1}, \xi_2^{a_2})$ and $\Pi_2(\xi_1^{a_1}, \xi_2^{a_2})$ are linearly independent. This can be ensured simply by renumbering the polynomials because, clearly, then the condition is equivalent to the requirement that $\Pi_1(\xi_1, \xi_2)$ and $\Pi_2(\xi_1, \xi_2)$ be linearly independent. Yet, if the number of variables is at least 3, the proof of Lemma 2.3 is somewhat bulky though quite elementary. So, we also postpone it till Section 3, see Subsection 3.2. At the moment, we restrict ourselves to the mere description of the operations to be used in order to modify the collection T. They will be of two types.

Type I. Passage to a complemented subspace. Let $t_1, t_2, \ldots, t_n$ be positive integers. Consider the space $\overline{X}$ of functions f on $\mathbb{T}^n$ such that $(z_1, \ldots, z_n) \mapsto f(z_1^{t_1}, \ldots, z_n^{t_n})$ is a function in $X = C^T(\mathbb{T}^n)$. Clearly, $\overline{X}$ is isomorphic to the subspace Y of X that consists of functions $g \in X$ such that $\widehat{g}(b_1, \ldots, b_n) = 0$ unless b_j is an integral multiple of t_j for all $j = 1, \ldots, n$. It is also clear that Y is complemented in X: a projection is given by convolution with an appropriate measure on $\mathbb{T}^n$. So, *it suffices to prove that Y^{**} does not embed complementedly in a $C(K)$-space.*

On the other hand, it is easy to realize that $\overline{X}$ is identifiable with $C^{\overline{T}}(\mathbb{T}^n)$, where $\overline{T} = \{\overline{T}_1, \dots, \overline{T}_J\}$, and each $\overline{T}_j$ is obtained from T_j by the formal substitution of the "variable" $t_i\partial_i$ for ∂_i. (Thus, $\overline{T}_j$ consists of the same differential monomials as T_j, but with other coefficients.) We see that the collection of the senior parts of the operators $\overline{T}_j$ also contains at least two linearly independent operators. So, we can pass to $\overline{T}$ from T whenever convenient.

Type II. Passage to linear combinations. Since the space $C_0^T(\mathbb{T}^n)$ depends in fact only on the linear span of the collection T, we may always replace the operators T_j by some other finite collection of operators generating the same linear space. Clearly, the senior parts of such a collection cannot be multiples of one of them, so this procedure is legitimate.

Lemma 2.3 will be proved in Subsection 3.2 by a combination of operations of these two types, after a special choice of the positive integers $t_1, t_2, \dots, t_n$ mentioned in the description of the first of them. Now we take the lemma for granted, but supplement it with yet another operation of type 2. Specifically, we denote $A(\xi_1, \xi_2) = \Pi_1(\xi_1, \xi_2^{a_2}, \dots, \xi_2^{a_n})$ and $B(\xi_1, \xi_2) = \Pi_2(\xi_1, \xi_2^{a_2}, \dots, \xi_2^{a_n})$. It has already been noted (see Remark 2.4) that these two polynomials are linearly independent. Let u be the greatest among the exponents x such that a nonzero monomial $c\xi_1^x\xi_2^v$, $(a_1x + v = k)$ occurs in A or in B. Interchanging, if necessary, the roles of Π_1 and Π_2, we may assume that a nonzero multiple of $\xi_1^u\xi_2^v$ occurs in A. Then we subtract a multiple of T_1 from T_2 in such a way that the term $\xi_1^u\xi_2^v$ disappear from B. Next, we normalize T_1 so that this term occur in A with coefficient 1. Afterwards, we denote by $u_1 < u$ the greatest exponent y such that B involves a nonzero monomial of the form $d\xi_1^y\xi_2^w$, $a_1y + w = k$. By simple manipulations as above, we can arrange that $d = 1$ and expel such a monomial from A if it occurs there.

2.3.1. A summary

For the reader's convenience, we summarize the agreements made up to this moment.

- The admissible hyperplane L we work with has the equation $\langle a, u\rangle = k$, where both k and the components a_j of the vector a are positive integers. Next, L intersects all positive coordinate semiaxes at integral points, so the quotients $m_j = k/a_j$ are also positive integers. Moreover, $a_1 \geqslant a_2 \geqslant a_3 \geqslant \dots \geqslant a_n > 0$.
- The collection $T_1, \dots, T_J$ of (differential) polynomials has the following properties. Let $\sigma_1, \dots, \sigma_J$ be the senior parts of the above polynomials with respect to L, and let Π_j be the characteristic polynomial for σ_j^*. Then the polynomials $A(\xi_1, \xi_2) = \Pi_1(\xi_1, \xi_2^{a_2}, \dots, \xi_2^{a_n})$ and $B(\xi_1, \xi_2) = \Pi_2(\xi_1, \xi_2^{a_2}, \dots, \xi_2^{a_n})$ in two variables are linearly independent. Furthermore, they have the following structure.
  - If we arrange the terms of A and B in the order of decrease of the exponent of ξ_1, then A starts with $\xi_1^u\xi_2^v$ and B starts with $\xi_1^{u_1}\xi_2^{v_1}$, where $u > u_1$;
  - A does not involve a multiple of the senior term of B;
  - $a_1u + v = a_1u_1 + v_1 = k$.

- We want to prove that $C_0^T(\mathbb{T}^n)^{**}$ does not embed complementedly in a $C(K)$-space under the above assumptions.

2.4. Construction of special elements in the annihilator of $C_0^T(\mathbb{T}^n)$ (with a modified collection T)

From now on we assume that the collection T has been modified as it was explained in the preceding subsections. We shall use some notation introduced there, in particular, the symbols A, B, u, and u_1 will retain their meaning.

We embed the space $C^T(\mathbb{T}^n)$ isomorphically in the direct sum $C(\mathbb{T}^n)\bigoplus\ldots\bigoplus C(\mathbb{T}^n)$ of J copies of $C(\mathbb{T}^n)$ by the mapping

$$f\mapsto (T_1f,\ldots,T_Jf).$$

The annihilator of the image of $C^T(\mathbb{T}^n)$ under this embedding consists of the collections $(\mu_1,\ldots,\mu_J)$ of measures on $\mathbb{T}^n$ such that

$$T_1^*\mu_1+\ldots+T_J^*\mu_J=0. \tag{2.4}$$

However, we work with the smaller space $C_0^T(\mathbb{T}^n)$ whose annihilator is determined by the condition

$$\sum_{j=1}^{J}\langle f,\, T_j^*\mu_j\rangle=0 \tag{2.5}$$

for all proper trigonometric polynomials f on $\mathbb{T}^n$. But observe that if the measures μ_j are proper themselves, then conditions (2.4) and (2.5) are equivalent. In this subsection we shall work with proper measures only, so we may use condition (2.4).

More specifically, in the annihilator of $C_0^T(\mathbb{T}^n)$ we shall construct certain elements of the form

$$\lambda\cdot\big(z_1^pz_2^qz_3^{[q^{\frac{a_3}{a_2}}]}\ldots z_n^{[q^{\frac{a_n}{a_2}}]},c_{pq}z_1^pz_2^qz_3^{[q^{\frac{a_3}{a_2}}]}\ldots z_n^{[q^{\frac{a_n}{a_2}}]},0,\ldots,0\big). \tag{2.6}$$

Here λ is the normalized Lebesgue measure on the torus. The positive integers p and q will be subject to certain conditions to be indicated later. In particular, all admissible p and q will be sufficiently large. Finally, the square brackets denote the integral part of a number, i.e., the greatest integer that does not exceed this number.

We shall ensure that the numbers c_{pq} be uniformly bounded. Then the elements (2.6) form a weakly 2-summable sequence in the annihilator because

$$\big\{z_1^pz_2^qz_3^{[q^{\frac{a_3}{a_2}}]}\ldots z_n^{[q^{\frac{a_3}{a_2}}]}\big\}_{p,q}$$

is an orthonormal system in $L^2(\lambda)$ (due to the first two factors, z_1^p and z_2^q). As has already been said, subsequently we shall exhibit a bounded linear operator from the annihilator to a Hilbert space that takes this sequence to a sequence that is not 2-summable.

Remark 2.5. If the number of variables is 2, we look for a collection

$$\lambda \cdot (z_1^p z_2^q, c_{pq} z_1^p z_2^q, 0, 0, \ldots, 0)$$

in place of (2.6), again with uniformly bounded c_{pq}. Such a collection can be found in some way as described below, but the calculations will simplify considerably.

So, we must find a collection of the form (2.6) that satisfies (2.4). Let P_1 and P_2 be the characteristic polynomials of the operators T_1^* and T_2^*, then (2.4) turns into

$$P_1(p, q, [q^{\frac{a_3}{a_2}}], \ldots, [q^{\frac{a_n}{a_2}}]) + c_{pq} P_2(p, q, [q^{\frac{a_3}{a_2}}], \ldots, [q^{\frac{a_n}{a_2}}]) = 0,$$

where

$$-c_{pq} = \frac{P_1(p, q, [q^{\frac{a_3}{a_2}}], \ldots, [q^{\frac{a_n}{a_2}}])}{P_2(p, q, [q^{\frac{a_3}{a_2}}], \ldots, [q^{\frac{a_n}{a_2}}])}.$$

A monomial of the form $ap^\alpha q^\beta$ (a is a numerical coefficient, α and β are not necessarily positive integers) is said to be *senior* if $a_1\alpha + a_2\beta = k$, and *junior* if $a_1\alpha + a_2\beta < k$. Observe the following elementary inequality (in it, $x_1, x_2, \ldots, x_n \geqslant 0$):

$$\begin{aligned} \Big|[x_1][x_2]\ldots[x_s] - x_1x_2\ldots x_s\Big| &\leqslant \big|[x_1] - x_1\big| \cdot [x_2][x_3]\ldots[x_s] + x_1\big|[x_2]\ldots[x_s] - x_2\ldots x_s\big| \\ &\leqslant x_2\ldots x_s + x_1\big|[x_1]\ldots[x_s] - x_2\ldots x_s\big|. \end{aligned}$$

Repeating the argument several times, we finally arrive at

$$\begin{aligned} \big|[x_1][x_2]\ldots[x_s] - x_1x_2\ldots x_s\big| \leqslant x_2\ldots x_s + x_1x_3x_4\ldots x_s + \\ x_1x_2x_4\ldots x_s + x_1\ldots x_{s-1} \end{aligned}$$

(each summand on the right involves $s-1$ factors). It follows that, for $l = 1, 2$, the quantity

$$|P_l(p, q, q^{\frac{a_3}{a_2}}, \ldots, q^{\frac{a_n}{a_2}}) - P_l(p, q, [q^{\frac{a_3}{a_2}}], \ldots, [q^{\frac{a_n}{a_2}}])|$$

is dominated by the sum of several junior monomials. Indeed, if $\alpha_1, \ldots, \alpha_n$ are natural numbers and $a_1\alpha_1 + \ldots + a_n\alpha_n \leqslant k$, then

$$p^{\alpha_1} q^{\alpha_2} q^{\alpha_3 \frac{a_3}{a_2}} \ldots q^{\alpha_n \frac{a_n}{a_2}} = p^{\alpha_1} q^{\frac{a_2\alpha_2 + \ldots + a_n\alpha_n}{a_2}}$$

is a monomial that definitely becomes junior if we drop any factor on the left.

We recall the notation A, B for the polynomials of two variables that arise from the senior parts of the polynomials P_1 and P_2 (see the end of the preceding subsection). Then we can write

$$|c_{pq}| \leqslant \frac{|A(p, q^{\frac{1}{a_2}})| + (\text{the absolute values of junior monomials})}{|B(p, q^{\frac{1}{a_2}})| - (\text{the absolute values of junior monomials})}$$
$$\leqslant \frac{p^u q^{\frac{v}{a_2}} + |\sum_{y<u, a_1 y + z = k} c_y p^y q^{\frac{z}{a_2}}| + \text{junior terms}}{p^{u_1} q^{\frac{v_1}{a_2}} - |\sum_{y<u_1, a_1 y + z = k} c'_y p^y q^{\frac{z}{a_2}}| - \text{junior terms}}.$$

Here the nonnegative integers u and u_1 are the same as at the end of the preceding subsection. Recall that $a_1 u + v = a_1 u_1 + v_1 = k$. It is convenient to rewrite the above inequality in terms of the “variables” $p^{\frac{1}{a_1}}$ and $q^{\frac{1}{a_2}}$:

$$|c_{pq}| \leqslant \frac{\big(p^{\frac{1}{a_1}}\big)^{a_1 u}\big(q^{\frac{1}{a_2}}\big)^{k-a_1 u} + \sum_{y<u} |c_y| \big(p^{\frac{1}{a_1}}\big)^{a_1 y}\big(q^{\frac{1}{a_2}}\big)^{k-a_2 y} + \text{junior terms}}{\big(p^{\frac{1}{a_1}}\big)^{a_1 u_1}\big(q^{\frac{1}{a_2}}\big)^{k-a_1 u_1} - \sum_{y<u_1} |c'_y| \big(p^{\frac{1}{a_1}}\big)^{a_1 y}\big(q^{\frac{1}{a_2}}\big)^{k-a_2 y} - \text{junior terms}} =$$
$$= \frac{\Big(\frac{p^{\frac{1}{a_1}}}{q^{\frac{1}{a_2}}}\Big)^{a_1 u} + \sum_{y<u} |c_y| \Big(\frac{p^{\frac{1}{a_1}}}{q^{\frac{1}{a_2}}}\Big)^{a_1 y} + \text{junior terms}}{\Big(\frac{p^{\frac{1}{a_1}}}{q^{\frac{1}{a_2}}}\Big)^{a_1 u_1} - \sum_{y<u_1} |c'_y| \Big(\frac{p^{\frac{1}{a_1}}}{q^{\frac{1}{a_2}}}\Big)^{a_1 y} - \text{junior terms}}.$$

Each “junior term” in the last expression is of the form

$$D \frac{(p^{1/a_1})^{\alpha}(q^{1/a_2})^{\beta}}{(p^{1/a_2})^k},$$

where $\alpha + \beta < k$ and $D > 0$. We impose the following condition on p and q: $\frac{\Delta}{2} \leqslant \frac{p^{1/a_1}}{q^{1/a_2}} \leqslant \Delta$, where Δ is a sufficiently large number. This number can be chosen so large that both in the numerator and in the denominator the middle term (the sum) becomes smaller than 1/10 of the first term. Next, every junior term satisfies the estimate

$$\frac{(p^{1/a_1})^{\alpha}(q^{1/a_2})^{\beta}}{(p^{1/a_2})^k} \leqslant \Delta^{\alpha} \frac{1}{(q^{1/a_2})^{k-\alpha-\beta}}.$$

We impose the condition $q > C$, where C is sufficiently large (C is chosen after Δ is fixed). Increasing C, we can make the sum of all junior terms (both in the numerator and in the denominator) smaller than 1/10 of the first term. So, the c_{pq} become uniformly bounded (in terms of Δ and C), provided p and q satisfy the above conditions.

To summarize: we have constructed a sequence of the form (2.6) in the annihilator of $C_0^T(\mathbb{T}^n)$ in such a way that the coefficients c_{pq} are uniformly bounded. This sequence is weakly 2-summable. The indices p and q range through arbitrary positive integers subject to the conditions $q > C$ and $\frac{\Delta}{2} \leqslant \frac{p^{1/a_1}}{q^{1/a_2}} \leqslant \Delta$, where Δ and C are sufficiently large fixed numbers.

2.5. An operator from the annihilator to a Hilbert space

Let $(\mu_1, \dots, \mu_J)$ be a collection of measures on the torus orthogonal to $C_0^T(\mathbb{T}^n)$ in the sense of condition (2.5). In the long run, we associate an element of a Hilbert space with the collection. This element will depend on the collection in a linear and continuous way, and will be constructed in several steps.

At the first step, we eliminate the terms $z_1^{k_1} \dots z_n^{k_n}$ with $\min_i |k_i| \leqslant 1000$ from the Fourier series of the μ_j. This is done by taking the 1001st remainder of the Fourier series with respect to all variables one after another. This linear and continuous operation results in a new collection of measures, still satisfying (2.5). To simplify the notation, we assume that the measures μ_j themselves satisfy $\widehat{\mu}(k_1, \dots, k_n) = 0$ if $|k_s| \leqslant 1000$ for at least one s. Such measures are proper automatically, so that condition (2.4) is fulfilled:

$$T_1^* \mu_1 + \dots + T_J^* \mu_J = 0.$$

Let ψ be a nonnegative infinitely differentiable function on $\mathbb{R}^n$ with $0 \leqslant \psi \leqslant 1$ everywhere, $\psi(\xi) = 0$ for $\max_j |\xi_j| > 20$, and $\psi(\xi) = 1$ for $\max_j |\xi_j| \leqslant 10$. We put $\varphi = \check{\psi} \in S(\mathbb{R}^n)$ and extend the measures μ_j to $\mathbb{R}^n$ periodically. Then the $\lambda_j = \varphi\mu_j$, $j = 1, \dots, J$, are finite measures on $\mathbb{R}^n$. We want to include the measures λ_j in an equation similar to (2.4).

For two multiindices α and β, the inequality $\beta \leqslant \alpha$ is understood coordinatewise; next, $\beta < \alpha$ means that $\beta \leqslant \alpha$ and $\beta \neq \alpha$.

Lemma 2.6. *Let α be a nonnegative multiindex, u a function in the Schwartz class, and λ a measure of temperate growth on $\mathbb{R}^n$ (i.e., every function in the Schwartz class is λ-integrable). Then the distribution $D^\alpha(u\lambda)$ is representable in the form*

$$D^\alpha(u\lambda) = uD^\alpha\lambda + \sum_{\beta \leq \alpha} c_\beta D^\beta \lambda_\beta, \tag{2.7}$$

where each λ_β is a measure obtained from λ by multiplication by some derivative of u, and the c_β are numerical coefficients.

The proof is quite simple and will be presented in Section 3 for completeness (see Subsection 3.3).

Now, since (2.4) is true, by Lemma 2.6 we see that

$$T_1^*(\varphi\mu_1) + \dots + T_J^*(\varphi\mu_J) = \varphi \cdot \sum_{j=1}^{J} T_j^* \mu_j + \sum_{\langle a,\alpha\rangle < k} b_\alpha D^\alpha \lambda_\alpha =$$

$$= \sum_{\langle a,\alpha\rangle < k} b_\alpha D^\alpha \lambda_\alpha,$$

where the λ_α are certain finite measures on $\mathbb{R}^n$ that depend linearly and continuously on the measures $\mu_1, \dots, \mu_J$ on $\mathbb{T}^n$, and the b_α are numerical coefficients. We recall that $T_j^* =$

$\sigma_j^* + \tau_j^*$, where σ_j^* is the senior part (see Subsection 2.3). So, finally we arrive at the equation

$$\sum_{j=1}^{J} \sigma_j^*(\varphi\mu_j) + \text{junior terms} = 0. \tag{2.8}$$

Each junior term is proportional to the derivative of order α with $\langle a, \alpha\rangle < k$ of a certain finite measure on $\mathbb{R}^n$ (and all these measures depend on $\mu_1, \dots, \mu_J$ linearly and continuously).

Our next goal is to get rid of the junior terms. This will be done with the help of a multiplier theorem. The Fourier transform of any derivative of φ is supported on the cube centered at zero and with edge length 40. Next, on the torus, the measures μ_j have no spectrum at least in the cube centered at zero and with edge length 2000. So, definitely, the Fourier transforms of the measures (on $\mathbb{R}^n$ this time) involved in the junior terms vanish in the ball $\{\xi\colon |\xi| \leqslant 100\}$. This will enable us to apply the following statement about multipliers. In it, the symbol L^1_{null} stands for the subspace of integrable functions on $\mathbb{R}^n$ whose Fourier transforms vanish on the set

$$\{\xi \in \mathbb{R}^n\colon\ |\xi| \leqslant \frac{1}{2}\}. \tag{2.9}$$

Proposition 2.7. *Let φ be a C^∞-function on $\mathbb{R}^n \setminus \{0\}$. Suppose that*

$$\varphi(t^{a_1}\xi_1, t^{a_2}\xi_2, \dots, t^{a_n}\xi_n) = t^{-\gamma}\varphi(\xi_1, \xi_2, \dots, \xi_n), \quad t > 0, \tag{2.10}$$

for some positive numbers $a_1, a_2, \dots, a_n, \gamma$. Then the linear operator M_φ defined by the formula

$$M_\varphi[f] = \mathcal{F}^{-1}\big[\varphi\mathcal{F}[f]\big]$$

on the set of Schwartz functions f with $\hat{f}$ vanishing on the set (2.9) extends to a bounded operator from the space L^1_{null} to itself.

Surely, the measures whose spectrum does not intersect the set (2.9) are also taken to measures by any multiplier as indicated in the above statement.

This purely technical proposition will also be proved in the next section, see Subsection 3.4. Now we are going to use it. We remind the reader that the admissible hyperplane we deal with is given by the equation $\langle a, \alpha\rangle = k$, where the (integral) coordinates of a are subject to certain conditions and (see Subsection 2.3) the numbers $m_j = k/a_j$ are positive integers. Up to a numerical coefficient, each "junior term" in (2.8) has the form $D^\beta\eta$, where η is a finite measure on $\mathbb{R}^n$ whose spectrum does not intersect the set (2.9) and $\langle a, \beta \rangle < k$, i.e. $\frac{\beta_1}{m_1} + \ldots + \frac{\beta_n}{m_n} < 1$. Consider the multiplier R_j whose symbol is

$$u_j(\xi_1, \ldots, \xi_n) = \frac{(2\pi i \xi_1)^{\beta_1} \ldots (2\pi i \xi_{j-1})^{\beta_{j-1}} (2\pi i \xi_j)^{\beta_j + 3m_j} (2\pi i \xi_{j+1})^{\beta_{j+1}} \ldots (2\pi i \xi_n)^{\beta_n}}{(2\pi i \xi_1)^{4m_1} + \ldots + (2\pi i \xi_n)^{4m_n}}. \tag{2.11}$$

(Note that the denominator vanishes only at zero.) The substitution

$$\xi \mapsto (t^{\frac{1}{4m_1}} \xi_1, \ldots, t^{\frac{1}{4m_n}} \xi_n)$$

is equivalent to multiplication of the symbol by

$$t^{-1} t^{\frac{1}{4}(\frac{\beta_1}{m_1} + \ldots + \frac{\beta_n}{m_n}) + \frac{3}{4}} \stackrel{\text{def}}{=} t^{\gamma},$$

and we see that $\gamma < 0$. By Proposition 2.7 and the remark after it, R_j takes η to a measure. On the other hand, clearly

$$D^{\beta} \eta = \sum_{j=1}^{n} \partial_j^{m_j} R_j \eta,$$

and all differential monomials $\partial_j^{m_j}$ are "senior" (correspond to multiindices lying on the affine hyperplane in question). Thus, equation (2.8) takes the form

$$\sum_{j=1}^{J} \sigma_j^*(\varphi \mu_j) + \sum_{j=1}^{n} \partial_j^{m_j} \omega_j = 0, \tag{2.12}$$

where the ω_j are certain finite measures on $\mathbb{R}^n$ that depend linearly and continuously on the initial measures $\mu_1, \ldots, \mu_J$. It is important that the measures ω_j have been obtained from some other measures by application of the multipliers described above. Next, the operators σ_j^* do not involve any of the differential monomial $\partial_i^{m_i}$ (this was ensured in the course of the modification of these operators, see Subsections 2.3). Collecting together all terms that correspond to any fixed differential monomial D^{α} (with $\langle a, \alpha \rangle = k$) in the first sum in (2.12), we arrive at the equation

$$\sum_{\langle a, \alpha \rangle = k} D^{\alpha} \nu_{\alpha} = 0, \tag{2.13}$$

where $\nu_{\alpha} = \omega_j$ if $\alpha = (0, 0, \ldots, m_j, 0, \ldots, 0)$ (m_j stays at jth place), and for other α the measure ν_{α} is a linear combination of the measures $\varphi \mu_j$ (later, we shall need some more details about the structure of this linear combination). In particular, the ν_{α} depend linearly and continuously on the initial collection $\mu_1, \ldots, \mu_J$.

We rewrite (2.13) in terms of Fourier transforms and restrict the result to the set of points of the form $(\xi_1, |\xi_2|, |\xi_2|^{a_3/a_2}, \ldots, |\xi_2|^{a_n/a_2})$:

$$\sum_{\langle a,\alpha\rangle=k}(2\pi i\xi_1)^{\alpha_1}(2\pi i)^{\alpha_2+\ldots+\alpha_n}|\xi_2|^{\frac{\alpha_2a_2+\ldots+\alpha_na_n}{a_2}}\widehat{\nu}_\alpha\big(\xi_1,|\xi_2|,|\xi_2|^{a_3/a_2},\ldots,|\xi_2|^{a_n/a_2}\big)=0,$$

or, equivalently,

$$\sum_{\langle a,\alpha\rangle=k}\xi_1^{\alpha_1}|\xi_2|^{\frac{k-a_1\alpha_1}{a_2}}\Big[(2\pi i)^{\alpha_1+\ldots+\alpha_n}\widehat{\nu}_\alpha\big(\xi_1,|\xi_2|,|\xi_2|^{a_3/a_2},\ldots,|\xi_2|^{a_n/a_2}\big)\Big]=0.$$

Finally, we put

$$f_s(\xi_1,\xi_2)=\sum_{\substack{\langle a,\alpha\rangle=k,\\ \alpha_1=s}}(2\pi i)^{\alpha_1+\ldots+\alpha_n}\widehat{\nu}_\alpha(\xi_1,|\xi_2|,|\xi_2|^{a_3/a_2},\ldots,|\xi_2|^{a_n/a_2}),$$

where we agree that the sum equals 0 if there are no α's such that $\alpha_1=s$ and $\langle\alpha,a\rangle=k$. Then the last equation takes the form

$$\sum_{s=0}^{m_1}\xi^s|\xi_2|^{\frac{(m_1-s)a_1}{a_2}}f_s(\xi_1,\xi_2)=0 \tag{2.14}$$

(note that the function f_s may have nonzero values only if $a_1s\leqslant k$, i.e., $s\leqslant k/a_1=m_1$).

It is important that f_s is the restriction of the Fourier transform of some finite measure on $\mathbb{R}^n$ to the "surface" $(\xi_1,\xi_2)\mapsto(\xi_1,|\xi_2|,|\xi_2|^{a_3/a_2},\ldots,|\xi_2|^{a_n/a_2})$.

We want to show that (2.14) is the solvability condition for a certain system of equations. Specifically, we claim that if (2.14) is fulfilled with continuous $f_0,f_1\ldots,f_{m_1}$ (in our setting, so they are *a priori*), then there exist function $\psi_1,\ldots,\psi_{m_1}$ in two variables that are continuous everywhere expect the origin and satisfy

$$\begin{cases}\xi_1\psi_1(\xi) = f_0(\xi);\\ \xi_1\psi_2(\xi)-|\xi_2|^{\frac{a_1}{a_2}}\psi_1(\xi) = f_1(\xi);\\ \quad\vdots \qquad = \qquad \vdots;\\ \xi_1\psi_j(\xi)-|\xi_2|^{\frac{a_1}{a_2}}\psi_{j-1}(\xi) = f_{j-1}(\xi);\\ \quad\vdots \qquad = \qquad \vdots;\\ \xi_1\psi_{m_1}(\xi)-|\xi_2|^{\frac{a_1}{a_2}}\psi_{m_1-1}(\xi)=f_{m_1-1}(\xi);\\ \qquad -|\xi_2|^{\frac{a_1}{a_2}}\psi_{m_1}(\xi) = f_{m_1}(\xi).\end{cases} \tag{2.15}$$

To show this, we note that, since the f_j are continuous, we can do the calculations outside the coordinate hyperplanes. We proceed by induction on m_1. If $m_1=1$, equation (2.14) turns into $|\xi|^{\frac{m_1a_1}{a_2}}f_0(\xi)+\xi_1f_1(\xi)=0$, i.e., $\frac{f_0(\xi)}{\xi_1}=-\frac{f_1(\xi)}{|\xi_2|^{\frac{m_1a_1}{a_2}}}=\psi_1(\xi)$, and we obtain the system

$$\begin{cases} -|\xi_2|^{\frac{m_1 a_1}{a_2}} \psi_1(\xi) = f_1(\xi), \\ \xi_1 \psi_1(\xi) = f_0(\xi). \end{cases} \tag{2.16}$$

The definition of ψ_1 (or, alternatively, system (2.16)) implies that ψ_1 is continuous everywhere except, maybe, the point 0. Thus, equation (2.14) with $m_1 = 1$ guarantees the solvability of (2.16) with respect to the "unknown" function ψ_1. This constitutes the base of induction.

Next, suppose we have proved the above claim for $m_1 - 1$. Then we rewrite (2.14) in the form

$$\psi_1(\xi) \stackrel{\text{def}}{=} \frac{f_0(\xi)}{\xi_1} = -\frac{\sum\limits_{s=1}^{m_1} \xi_1^{s-1} |\xi_2|^{\frac{(m_1-s)a_1}{a_2}} f_s(\xi)}{|\xi_2|^{\frac{m_1 a_1}{a_2}}} =$$

$$= -\frac{\sum\limits_{s=0}^{m_1-1} \xi_1^{s} |\xi_2|^{\frac{(m_1-1-s)a_1}{a_2}} f_{s+1}(\xi)}{|\xi_2|^{\frac{m_1 a_1}{a_2}}}.$$

Thus, we have obtained the following two identities:

$$\xi_1 \psi_1(\xi) = f_0(\xi),$$

$$|\xi_2|^{\frac{(m_1-1)a_1}{a_2}} \left(f_1(\xi) + |\xi_2|^{\frac{a_1}{a_2}} \psi_1(\xi)\right) + \sum_{s=0}^{m_1-1} \xi_1^s |\xi_2|^{\frac{(m_1-1-s)a_1}{a_2}} f_{s+1}(\xi) = 0,$$

the second of which is equation (2.14) (with m_1 replaced by $m_1 - 1$) for the functions

$$f_1(\xi) + |\xi_2|^{\frac{a_1}{a_2}} \psi_1(\xi), f_2(\xi), \dots, f_{m_1}(\xi).$$

Applying the inductive hypothesis, we prove the claim for the value m_1.

Now, *we summarize*: the required operator from the annihilator of the space $C_0^T(\mathbb{T}^n)$ to a Hilbert space acts by the rule

$$(\mu_1, \dots, \mu_J) \mapsto (f_0, \dots, f_{m_1}) \mapsto (\psi_1, \dots, \psi_{m_1}).$$

The Hilbert space in question is the weighted L^2 on the plane that arises when we specify the parameters in Theorem 1.1 as follows: $\theta_j = a_j/a_2$ for $j \geqslant 2$ and $\eta = a_1/a_2$. For the reader's convenience, we restate the required fact.

Theorem 2.8. *Let equations (2.15) be fulfilled, where*

$$f_j(\xi_1, \xi_2) = \widehat{\gamma}_j(\xi_1, |\xi_2|, |\xi_2|^{a_3/a_2}, \dots, |\xi_2|^{a_n/a_2})$$

and the γ_j are finite measures on $\mathbb{R}^n$. Then

$$\max_j \Big(\int\limits_{\mathbb{R}^2} |\psi_j(\xi)|^2 |\xi_2|^{\frac{a_1}{a_2}-1} d\xi \Big)^{\frac{1}{2}} \lesssim \max_j \|\gamma_j\|.$$

2.6. Contradiction

As was already said in the introductory Subsection 2.1, by the Grothendieck theorem, if $C_0^T(\mathbb{T}^n)^{**}$ is complemented in a space of type $C(K)$, then the operator constructed in the preceding subsection must be 1-summing and, *a fortiori*, 2-summing. In Subsection 2.4, we constructed special elements of the form

$$\lambda(z_1^p z_2^q z_3^{[q^{\frac{a_3}{a_2}}]} \dots z_n^{[q^{\frac{a_n}{a_2}}]}, c_{pq} z_1^p z_2^q z_3^{[q^{\frac{a_3}{a_2}}]} \dots z_n^{[q^{\frac{a_n}{a_2}}]}, 0, \dots, 0) \tag{2.17}$$

in the annihilator of $C_0^T(\mathbb{T}^n)$. We remind the reader that these elements constitute a weakly 2-summable sequence, so we shall arrive at a contradiction if we show that the squares of the norms of their images under the operator constructed in Subsection 2.5 constitute a divergent series.

Recall that p and q were subject to certain restrictions implying, among other things, that p and q are sufficiently large. We may assume if necessary that they are even larger, in particular, we may require that all components of the vector $(p, q, [q^{\frac{a_3}{a_2}}], \dots, [q^{\frac{a_n}{a_2}}])$ be greater than 1000. Also, recall that λ in (2.17) denotes the Lebesgue measure on the torus. Let $\pi_{p,q}$ denote the measure $z_1^p z_2^q z_3^{[q^{\frac{a_3}{a_2}}]} \dots z_n^{[q^{\frac{a_n}{a_2}}]} \cdot \lambda$ on the torus $\mathbb{T}^n$, then the elements (2.17) become

$$(\pi_{p,q}, c_{pq}\pi_{p,q}, 0, \dots, 0). \tag{2.18}$$

They satisfy the equation

$$T_1^* \pi_{p,q} + c_{pq} T_2^* \pi_{p,q} = 0,$$

see equation (2.4). Let ψ be the function introduced at the beginning of Subsection 2.5, and let $\varphi = \check{\psi}$ as in that subsection. The Fourier transform (on $\mathbb{R}^n$) of the measure $\varphi\pi_{p,q}$ is the function

$$(\xi_1, \xi_2, \dots, \xi_n) \mapsto \psi\Big(\xi_1 - p, \xi_2 - q, \xi_3 - [q^{\frac{a_3}{a_2}}], \dots, \xi_n - [q^{\frac{a_n}{a_2}}]\Big). \tag{2.19}$$

In accordance with Subsection 2.5, we shall need the restriction of this function to the "surface"

$$(\xi_1, \xi_2) \mapsto (\xi_1, |\xi_2|, |\xi_2|^{\frac{a_3}{a_2}}, \dots, |\xi_2|^{\frac{a_n}{a_2}}).$$

We denote this restriction by $\varkappa_{p,q}$:

$$\varkappa_{p,q}(\xi_1, \xi_2) = \psi(\xi_1 - p, |\xi_2| - q, |\xi_2|^{\frac{a_3}{a_2}} - [q^{\frac{a_3}{a_2}}], \dots, |\xi_2|^{\frac{a_n}{a_2}} - [q^{\frac{a_n}{a_2}}]).$$

For $j \geqslant 3$ we have

$$\big||\xi_j|^{\frac{a_j}{a_2}} - \big[q^{\frac{a_j}{a_2}}\big]\big| \leqslant 1 + \big||\xi_j|^{\frac{a_j}{a_2}} - q^{\frac{a_j}{a_2}}\big| \leqslant 1 + \big||\xi_j| - q\big|$$

because $a_j \leqslant a_2$ for $j \geqslant 3$ (this agreement was adopted in Subsection 2.3). It follows that the relation $\varkappa_{p,q}(\xi_1, \xi_2) = 1$ is guaranteed by the inequalities $|\xi_1 - p| \leqslant 5$ and $||\xi_2| - q| \leqslant 5$. On the other hand, clearly, $\varkappa_{p,q}(\xi_1, \xi_2) = 0$ whenever $\max\{|\xi_1 - p|, ||\xi_2| - q|\} > 20$.

Now, in addition to all restrictions indicated above, we agree to consider only p and q so large that $\xi_1 \asymp p$ and $|\xi_2| \asymp q$ for every pair (ξ_1, ξ_2) with $\varkappa_{p,q}(\xi_1, \xi_2) \neq 0$. (Formula $a \asymp b$ means that the quotient of the quantities $|a|$ and $|b|$ is bounded and bounded away from zero.)

Taking the relation $q^{\frac{1}{a_2}} \asymp p^{\frac{1}{a_1}}$ into account (see Subsection 2.4), we obtain

$$|\xi_2|^{\frac{a_1}{a_2}} \asymp q^{\frac{a_1}{a_2}} \asymp p \asymp \xi_1 \tag{2.20}$$

for such pairs (ξ_1, ξ_2).

Now, we need to analyze what relations (2.12), (2.13), and (2.14) will look like in our situation. More precisely, we analyze the structure of the measures η_α in (2.13) and the functions f_s in (2.14). We start with (2.12). There are only two "initial" measures ($\pi_{p,q}$ and $c_{pq}\pi_{p,q}$) in the present setting, and they correspond to the indices $j = 1, 2$ in formula (2.12). That formula involves also the second sum composed of the terms $\partial_j^{m_j}\omega_j$. But all measures ω_j were obtained by application of certain multipliers whose symbols decay at infinity with a power-type rate. More precisely, if we put $\xi_j = |\xi_2|^{\frac{a_j}{a_2}}$, $j > 2$, in formula (2.11) for a multiplier's symbol and then use the fact that $\xi_1 \asymp |\xi_2|^{\frac{a_1}{a_2}} = |\xi_2|^{\frac{m_2}{m_1}}$ in the domain in question, we see that the symbol behaves as

$$\frac{|\xi_1|^{\beta_1}|\xi_2|^{m_2(\frac{\beta_2}{m_2}+\ldots+\frac{\beta_n}{m_n})+3m_2}}{|\xi_1|^{4m_1} + |\xi_2|^{4m_2}} \asymp \frac{|\xi_2|^{m_2(\frac{\beta_1}{m_1}+\ldots+\frac{\beta_n}{m_n})+3m_2}}{|\xi_2|^{4m_2}} \asymp$$
$$\asymp |\xi_2|^{m_2(-1+\frac{\beta_1}{m_1}+\ldots+\frac{\beta_n}{m_n})} \asymp |\xi_1|^{-m_1\sigma},$$

where $\sigma = 1 - (\frac{\beta_1}{m_1} + \ldots + \frac{\beta_n}{m_n}) > 0$.

Now, in (2.14) the function $f_0(\xi_1, \xi_2)$ arises from the second sum in (2.12), so that $f_0(\xi_1, \xi_2) = O(|\xi_1|^{-m_1\sigma})$, uniformly in p and q (note that we do not reflect the dependence of the function f_j on p and q in our notation). We also must examine the functions $f_1, \ldots, f_{u-1}, f_u$, where u was introduced at the end of Subsection 2.3. We recall the notation A and B for two special polynomials of two variables, also introduced at the end of that subsection. The point is that $f_1, \ldots, f_u$ arise from the expression

$$A(\xi_1, |\xi_2|^{\frac{1}{a_2}})\varkappa_{p,q}(\xi_1, \xi_2) + B(\xi_1, |\xi_2|^{\frac{1}{a_2}})c_{pq}\varkappa_{p,q}(\xi_1, \xi_2)$$

after we collect similar terms. In fact, they arise from the first summand (involving A), because the polynomial B does not involve terms of the form $C \cdot \xi_1^x \xi_2^v$ with $x \geqslant u$

and $C \neq 0$, by construction. Moreover, it is easy to realize that if $u > 1$ (note that the case of $u = 1$ may also occur), then $f_1, \dots, f_{u-1}$ are identically zero, while $f_u = \varkappa_{p,q}$ for any u. Thus, several first equations of system (2.15) look like this (now we do reflect the dependence of f_j on p and q):

$$\xi_1 \psi_1^{(p,q)}(\xi) = f^{(p,q)}(\xi),$$
$$-|\xi_2|^{\frac{a_1}{a_2}} \psi_j^{(p,q)}(\xi) + \xi_1 \psi_{j+1}^{(p,q)}(\xi) = 0,\ \ j = 1, \dots, u-1;$$
$$-|\xi_2|^{\frac{a_1}{a_2}} \psi_u^{(p,q)}(\xi) + \xi_1 \psi_{u+1}^{(p,q)}(\xi) = \varkappa_{p,q}(\xi).$$

There is a slight difference from the above if $u = 1$, which is not important. The other equations are not important at the moment either.

We resolve the above equations consecutively, starting with the first, to find all $\psi_j^{(p,q)}$. Recall that $f_0^{(p,q)}(\xi) = O(|\xi_1|^{-\varepsilon})$ uniformly in p and q, for some $\varepsilon > 0$, and that $\xi_1 \asymp |\xi_1|^{\frac{a_1}{a_2}}$ on the supports of the $\varkappa_j^{(p,q)}$. It is easy to deduce that

$$\psi_{u+1}^{(p,q)}(\xi) = \frac{1}{\xi_1}\Big(\varkappa_{p,q}(\xi) + O(|\xi_1|^{-\varepsilon})\Big);$$

moreover, $\psi_{n+1}^{(p,q)}$ may be nonzero only on the support of $\varkappa_{p,q}$. Assuming that p and q are sufficiently large, we can ensure that the second term in parentheses be smaller than, say, 10^{-6} on the support of $\varkappa_{p,q}$. Now, it becomes clear that

$$\int\limits_{\mathbb{R}^2} |\psi_{u+1}^{(p,q)}(\xi)|^2 |\xi_2|^{\frac{a_1}{a_2}-1} d\xi \gtrsim \iint\limits_{\substack{|\xi_1 - p| \leqslant 5 \\ ||\xi_2| - q| \leqslant 5}} |\xi_1|^{-2} |\xi_2|^{\frac{a_1}{a_2}-1} d\xi \geqslant$$
$$\gtrsim p^{-2} q^{\frac{a_1}{a_2}} q^{-1} \gtrsim p^{-2} p q^{-1} = p^{-1} q^{-1}$$

(we have used (2.20)).

Finally, we must sum $p^{-1}q^{-1}$ over all admissible p and q. For every q we have $p \asymp q^{\frac{a_1}{a_2}}$ and the number of admissible p is also roughly $q^{\frac{a_1}{a_2}}$. Thus,

$$\sum_q \sum_p p^{-1} q^{-1} \gtrsim \sum_q q^{\frac{a_1}{a_2}} (q^{-\frac{a_1}{a_2}} q^{-1}) = D \sum_q q^{-1} = \infty. \qquad \square$$

3. Proofs of auxiliary statements

In this section, we present the technical proofs that were omitted to speed up the development of the plot. With one exception, we turn to them in the order they were omitted.

3.1. Rotating the admissible hyperplane

Proof of Lemma 2.1. First, we note that the rationality of p_1 will follow if we ensure that $\tilde{L}$ contain n linearly independent integral vectors. Second, let $\mathcal{X}$ be the linear span of $\{X_1 - X_0, X_2 - X_0, \ldots, X_N - X_0\}$. Let $\ell_1, \ell_2, \ldots, \ell_s$ be vectors in $\mathbb{R}^n$ such that the vectors $p, \ell_1, \ell_2, \ldots, \ell_s$ form an orthonormal basis in $\mathcal{X}^\perp$. Let λ be a large number. The set $\mathbb{Z}^n$ forms a $\sqrt{n}$-net in $\mathbb{R}^n$, therefore, for each s there exists a point $E_s \in \mathbb{Z}^n$ such that $\operatorname{dist}(\lambda\ell_s, E_s) \leqslant \sqrt{n}$. Define the hyperplane $\tilde{L}$ to be the affine span of $X_0, X_1, \ldots, X_N$ and $X_0 + E_1, X_0 + E_2, \ldots, X_0 + E_s$. This hyperplane contains at least n integral points (because $N \geqslant \dim \mathcal{X} = n - s - 1$). If λ is sufficiently large, then these points are linearly independent. Let p_1 be a vector such that $\langle X, p_1\rangle = 1$ for all $X \in \tilde{L}$. The claim of the lemma will follow provided we prove that $p_1 \to p$ as $\lambda \to \infty$. The former vector is defined by the conditions

$$\langle X_i, p_1\rangle = 1, \quad i = 0, 1, \ldots, N; \quad \langle E_i, p_1\rangle = 0, \quad i = 1, 2, \ldots, s.$$

The second group of equations can be rewritten as $\langle \ell_i - \frac{\lambda\ell_i - E_i}{\lambda}, p_1\rangle = 0$. Thus, the matrix of the system that determines p_1 tends to the matrix of the system that determines p. The latter is nonsingular, so $p_1 \to p$. Therefore, $\langle Z_j, p_1\rangle < 1$ and p_1 is a vector with positive coordinates eventually. □

3.2. Modification of the collection T

Proof of Lemma 2.3. In Subsection 2.3, we described two types of admissible operations that can be done with the collection $T = \{T_j\}$ of differential expressions in question. We shall see that a series of such operations will provide us with a new collection satisfying the claim of Lemma 2.3.

Let us return to formula (2.3) and recall that the symbol Π_j on the right stands for the senior part of the characteristic polynomial P_j for the adjoint T_j^*. The coefficients of the polynomial Π_j will be denoted by $d_{\alpha,j}$.

We apply an operation of Type I (see Subsection 2.3) with parameters $t_1, t_2, \ldots, t_n$ (recall that they are positive integers), to pass to a new collection $\overline{T} = \{\overline{T}_1, \ldots, \overline{T}_J\}$ of differential operators. For this new collection, an analog of formula (2.3) looks like this:

$$\overline{P}_j(\xi_1, \ldots, \xi_n) = P_j(t_1\xi_1, \ldots, t_n\xi_n) = \Pi_j(t_1\xi_1, \ldots, t_n\xi_n) + \\ + \rho_j(t_1\xi_1 + \ldots + t_n\xi_n) \stackrel{\text{def}}{=} \overline{\Pi}_j(\xi_1, \ldots, \xi_n) + \overline{\rho}_j(\xi_1, \ldots, \xi_n).$$

Reindexing the P_j if necessary, we may assume that Π_1 is a nonzero polynomial. We impose the condition $\Pi_1(t_1, \ldots, t_n) \neq 0$ on the positive integers $t_1, \ldots, t_n$ (yet another condition on these numbers will be imposed later).

Recall (see Subsection 2.3) that at least two polynomials among the $\overline{\Pi}_j(\xi_1, \ldots, \xi_n)$ are linearly independent. Then two linearly independent objects also occur among the polynomials

$$\widetilde{\Pi}_j(\zeta_1, \dots, \zeta_n) = \overline{\Pi}_j(\zeta_1^{a_1}, \dots, \zeta_n^{a_n}),$$

because this change of variables takes the monomial $\xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$ to $\zeta_1^{\alpha_1 a_1} \dots \zeta_n^{\alpha_n a_n}$, and different monomials give rise to different ones. We introduce yet another series of variables:

$$\zeta_1 = \eta_1, \zeta_2 = \eta_1 + \eta_2, \dots, \zeta_n = \eta_1 + \eta_n.$$

This substitution is invertible, so two linearly independent objects occur also among the polynomials

$$\widetilde{\widetilde{\Pi}}_j(\eta_1, \dots, \eta_n) = \widetilde{\Pi}_j(\eta_1, \eta_1 + \eta_2, \dots, \eta_1 + \eta_n).$$

Next, we have

$$\widetilde{\widetilde{\Pi}}_j(\eta_1, \dots, \eta_n) = \Pi_j(t_1\eta_1^{a_1}, t_2(\eta_1 + \eta_2)^{a_2}, \dots t_n(\eta_1 + \eta_n)^{a_n}).$$

Expanding, we see that the coefficient of the term η_1^k in $\widetilde{\widetilde{\Pi}}_j(\eta_1, \dots, \eta_n)$ is equal to $\Pi_j(t_1, \dots, t_n)$. Indeed, there is only one way to obtain $\eta_1^k = \eta_1^{a_1\alpha_1 + \dots + a_n\alpha_n}$ when we expand the product

$$d_{\alpha,j}(t_1\eta_1^{a_1})^{\alpha_1}(t_2(\eta_1 + \eta_2)^{a_2})^{\alpha_2} \dots (t_n(\eta_1 + \eta_n)^{a_n})^{\alpha_n},$$

etc. By the above, this coefficient is nonzero for $j = 1$. We put

$$\gamma_j = \frac{\Pi_j(t_1, \dots, t_n)}{\Pi_1(t_1, \dots, t_n)}$$

for $j \geqslant 2$. Then the polynomials $\widetilde{\widetilde{\Pi}}_j(\eta_1, \dots, \eta_n) - \gamma_j\widetilde{\widetilde{\Pi}}_1(\eta_1, \dots, \eta_n)$ do not involve the term proportional to η_1^k for $j \geqslant 2$ and, moreover, at least one of these polynomials is nonzero. There is no loss of generality in assuming that $\widetilde{\widetilde{\Pi}}_2 - \gamma_2\widetilde{\widetilde{\Pi}}_1 \neq 0$.

Now, we modify the initial polynomials by an operation of Type 2 (see Subsection 2.3): P_1 is left as it is, and each P_j with $j \geqslant 2$ is replaced by $P_j - \gamma_j P_1$. This does not change the space $C^T(\mathbb{T}^n)$. For short, we redenote these new polynomials by the old symbols $P_1, \dots, P_J$ and assume that all the above construction have been done for the new polynomials (thus, some intermediate notation acquires a different meaning). In this new setting, it turns out that $\widetilde{\widetilde{\Pi}}_2(\eta_1, \dots, \eta_n)$ is a nonzero polynomial not involving the term proportional to η_1^k but $\widetilde{\widetilde{\Pi}}_1(\eta_1, \dots, \eta_n)$ involves η_1^k with a nonzero coefficients.

We want to ensure that the polynomial $\widetilde{\widetilde{\Pi}}_2(\eta_1, \eta_2, \eta_2, \dots, \eta_2)$ (in two variables) be nonzero. For this, we examine the expression that arises from one monomial involved in $\Pi_2(\eta_1, \ldots, \eta_n)$, this is

$$d_{\alpha,2} t_1^{\alpha_1} \ldots t_n^{\alpha_n} \eta_1^{a_1\alpha_1} (\eta_1 + \eta_2)^{a_2\alpha_2 + \ldots + a_n\alpha_n}.$$

We expand and choose the term in which η_2 occurs with the largest possible exponent. This term is equal to

$$d_{\alpha,2} t_1^{\alpha_1} \ldots t_n^{\alpha_n} \eta_1^{a_1\alpha_1} \eta_2^{k - a_1\alpha_1}.$$

Now, the exponent $k - a_1\alpha_1$ takes the maximal value if α_1 is the smallest possible. We denote $\overline{\alpha} = \min\{\alpha_1 \colon \langle a, \alpha\rangle = k, d_{\alpha,2} \neq 0\}$. Then, after expansion, the polynomial $\tilde{\tilde{\Pi}}_2(\eta_1, \eta_2, \ldots, \eta_2)$ contains the term

$$\eta_1^{a_1\overline{\alpha}} \eta_2^{k - a_1\overline{\alpha}} \sum_{\substack{\langle a,\alpha\rangle = k \\ d_{\alpha,2} \neq 0,\ \alpha_1 = \overline{\alpha}}} d_{\alpha,2} t_1^{\overline{\alpha}} t_2^{\alpha_2} \ldots t_n^{\alpha_n}.$$

The coefficient of the monomial $\eta_1^{a_1\overline{\alpha}} \eta_2^{k - a_1\overline{\alpha}}$ in this term is a nonzero polynomial p in the variables $t_1, \ldots, t_n$. We fix $t_1, t_2, \ldots, t_n$ such that $p(t_1, \ldots, t_n) \neq 0$ and (which was assumed from the outset) $\Pi_1(t_1, \ldots, t_n) \neq 0$.

After this choice, the polynomials $\tilde{\tilde{\Pi}}_1(\eta_1, \eta_2, \ldots, \eta_2)$ and $\tilde{\tilde{\Pi}}_2(\eta_1, \eta_2, \ldots, \eta_2)$ become linear independent. *A fortiori*, so are the polynomials

$$\overline{\Pi}_1(\xi_1^{a_1}, \xi_2^{a_2}, \ldots, \xi_2^{a_n}) \text{ and } \overline{\Pi}_2(\xi_1^{a_1}, \xi_2^{a_2}, \ldots, \xi_2^{a_n}).$$

This finishes the proof of Lemma 2.3. □

Remark 3.1. The role of the variables ζ_i and η_i was purely auxiliary, they are not used anywhere except the above proof.

3.3. Proof of Lemma 2.6

As has already been said, we include this elementary argument for completeness only. If $\alpha = 0$, there is nothing to prove. Suppose that (2.7) is true for some multiindex α. Differentiating that equation by the jth variable, we obtain

$$\partial_j D^\alpha(u\lambda) = u\partial_j D^\alpha\lambda + \partial_j u D^\alpha\lambda + \sum_{\beta<\alpha} c_\beta \partial_j D^\beta \lambda_\beta.$$

It remains to verify the following claim.

Sublemma. *If $v \in S^\infty(\mathbb{R}^n)$, then*

$$vD^\gamma\lambda = \sum_{\beta\leqslant\gamma} \alpha_\beta D^\beta \widetilde{\lambda}_\beta \tag{3.1}$$

with some numerical coefficients α_β, where the $\widetilde{\lambda}_\beta$ are measures of the same form as the measures λ_β above.

Again there nothing to prove for $\gamma = 0$. But if (3.1) is true for some γ and all v, we write

$$v\partial_j D^\gamma \lambda = \partial_j(vD^\gamma \lambda) - (\partial_j v)D^\gamma \lambda$$

and apply the inductive hypothesis (3.1) both to $vD^\gamma \lambda$ and to $(\partial_j v)D^\gamma \lambda$.

3.4. Multipliers on L_1

In this subsection, we prove Proposition 2.7. When we prepared the paper, we easily found a proof that required no background, the main tool in which was integration by parts, and the crucial estimate was related to the observation that a continuous analog of the Dirichlet kernel was involved, which still has an at most logarithmic growth. See also the somewhat similar Lemma 1.10 in [13] about L^1-multipliers on the torus. Naturally, summation by parts (instead of integration by parts) was used in its proof. But later we found a result in the literature (see Theorem 1.2 in [3]; see also [1] for the proof), which shortens calculations, and we prefer to use it. Here is the statement.

Theorem 3.2. *Suppose that φ belongs to $C(\mathbb{R}^n)$ and satisfies the conditions*

(1) *for all multiindices $\alpha \in \mathbb{Z}_2^n$ the classical derivatives $D^\alpha f$ exist and*

$$\lim_{|\xi_j|\to\infty} \partial^\alpha \varphi(\xi) = 0 \quad \text{for all} \quad j = 1, 2, \dots, n;$$

(2) *there exists a positive number δ such that*

$$\prod_{j=1}^{n} |\xi_j|^{1-\delta}(1 + |\xi_j|^{2\delta})\big|\partial_1\partial_2 \dots \partial_n \varphi(\xi)\big| \lesssim 1.$$

Then φ is a Fourier multiplier of $L^1(\mathbb{R}^n)$.

Proof of Proposition 2.7. Let η be some smooth function on $\mathbb{R}^n$ such that $\eta(\xi) = 1$ when $|\xi| \geqslant \frac{1}{3}$, but $\eta(\xi) = 0$ when $|\xi| < \frac{1}{4}$. It suffices to show that $\varphi\eta$ is a Fourier multiplier on $L^1(\mathbb{R}^n)$. We claim that this function satisfies the assumptions of Theorem 3.2. We note that the function φ satisfies the condition

$$t^{\alpha_j}\partial_j\varphi(t^{a_1}\xi_1, t^{a_2}\xi_2, \dots, t^{a_n}\xi_n) = t^{-\gamma}\partial_j\varphi(\xi_1, \xi_2, \dots, \xi_n), \tag{3.2}$$

which can be obtained from formula (2.10) by differentiation. Similar identities for higher order derivatives show that the first condition of Theorem 3.2 is satisfied. To verify the second condition, we use the identity

$$t^{\sum\limits_j \alpha_j} \partial_1\partial_2\dots\partial_n\varphi(t^{a_1}\xi_1, t^{a_2}\xi_2, \dots, t^{a_n}\xi_n) = t^{-\gamma}\partial_1\partial_2\dots\partial_n\varphi(\xi_1, \xi_2, \dots, \xi_n).$$

We take $t = \big(\sum_j |\xi_j|^{\frac{1}{\alpha_j}}\big)^{-1}$. Then the point $(t^{a_1}\xi_1, t^{a_2}\xi_2, \dots, t^{a_n}\xi_n)$ lies on the surface

$$\Big\{\xi \in \mathbb{R}^n\colon \sum_{j=1}^n |\xi_j|^{\frac{1}{\alpha_j}} = 1\Big\},$$

so φ at this point is bounded by some uniform constant. Thus,

$$\Big|\partial_1\partial_2\dots\partial_n\varphi(\xi_1, \xi_2, \dots, \xi_n)\Big| \lesssim \Big(\sum_j |\xi_j|^{\frac{1}{\alpha_j}}\Big)^{-\sum\limits_j \alpha_j - \gamma}.$$

We may assume that $1 \lesssim \sum\limits_j |\xi_j|^{\frac{1}{\alpha_j}}$. We must prove the inequality

$$\Big(\prod_{j=1}^n |\xi_j|^{1-\delta}(1 + |\xi_j|^{2\delta})\Big)\Big(\sum_j |\xi_j|^{\frac{1}{\alpha_j}}\Big)^{-\sum\limits_j \alpha_j - \gamma} \lesssim 1$$

provided δ is sufficiently small. Here is the proof (first, we use the boundedness from below of the sum, second, we use the Cauchy inequality for arithmetic and geometric means, third, we use the easy inequality $(1 + |\xi|^d)^{-\varepsilon} \lesssim (1 + |\xi|)^{-d\varepsilon}$ for $d, \varepsilon > 0$, and finally, we use the fact that $\frac{\sum_j \alpha_j + \gamma}{\sum_j \alpha_j} > 1$):

$$\Big(\prod_{j=1}^n |\xi_j|^{1-\delta}(1 + |\xi_j|^{2\delta})\Big)\Big(\sum_j |\xi_j|^{\frac{1}{a_j}}\Big)^{-\sum\limits_j a_j - \gamma} \lesssim$$

$$\Big(\prod_{j=1}^n |\xi_j|^{1-\delta}(1 + |\xi_j|^{2\delta})\Big)\Big(1 + \sum_j \frac{a_j}{\sum_j a_j}|\xi_j|^{\frac{1}{a_j}}\Big)^{-\sum\limits_j a_j - \gamma} \lesssim$$

$$\prod_{j=1}^n \Big(|\xi_j|^{1-\delta}(1 + |\xi_j|^{2\delta})\big(1 + |\xi_j|^{\frac{1}{a_j}}\big)^{\frac{-(\sum\limits_j a_j + \gamma)a_j}{\sum\limits_j a_j}}\Big) \lesssim$$

$$\prod_{j=1}^n \Big((1 + |\xi_j|)^{1+\delta}\big(1 + |\xi_j|\big)^{\frac{-(\sum\limits_j a_j + \gamma)}{\sum\limits_j a_j}}\Big) \lesssim 1. \qquad \square$$

3.5. A fact from the Banach space theory

Here we give some hints to the proof of an abstract statement used crucially in the paper (see the second paragraph in Subsection 2.1 and also the beginning of Subsection 2.6

about the way in which it was applied; see also the paragraph after Proposition 4.1 below, where the same statement will be involved once again).

Theorem. *Let Y be a Banach space and X its closed subspace. If X^{**} embeds complementedly in $C(K)$ for some compact space K, then the annihilator $X^\perp = \{f \in Y^*: f(x) = 0$ for all $x \in X\}$ is complemented in Y^*. In particular, if Y itself is a space of type $C(T)$, then $X^\perp$ verifies the Grothengieck theorem: every bounded linear operator from $X^\perp$ to a Hilbert space is 1-absolutely summing.*

Proof. We restrict ourselves to a cursory sketch, because the result is well known (but maybe not as well now as it was two or there decades ago). By Theorem 12 in [17], X^* is an $\mathcal{L}_1$-space. We remind the reader that this means that there is a family $\mathcal{F}$ of finite-dimensional subspaces of X^* directed by inclusion and such that $\sup_{F\in\mathcal{F}} \operatorname{dist}(F, l^1_{\dim F}) = \lambda < \infty$ and $\cup_{F\in\mathcal{F}} F = X^*$. Here dist stands for the Banach–Mazur distance. The notion of an $\mathcal{L}_1$-space was introduced in [16].

Now, consider the canonical quotient mapping $Q: Y^* \to X^*$. Since X^* is an $\mathcal{L}_1$-space, we are under the assumptions of Theorem 5.1 in [29], which yields a bounded linear section $S: X^* \to Y^*$ for Q (i.e., $QS = \mathrm{id}$). The required projection of Y^* onto $X^\perp$ acts by the rule $f \mapsto f - SQf$.

The second claim follows from the fact that the dual to $C(T)$ does verify the Grothendieck theorem (see [16] or [28]), and that this property is inherited by complemented subspaces. □

4. Beyond Theorem 0.1

In this section we want to present some information about the case where Theorem 0.1 is not applicable. Sometimes we can prove that its conclusion is still fulfilled, in some other cases we can prove that it fails.

4.1. Reduction of dimension and other examples

In this subsection we discuss some examples of nonisomorphism not covered by Theorem 0.1. We start with two simple observations. In the first, the answer is clear but indirect. Consider the collection T consisting of two operators $T_1 = \partial_1^2 + 2\partial_1\partial_2 + \partial_2^2$ and $T_2 = a\partial_1 + b\partial_2$ on the torus $\mathbb{T}^2$. It is obvious that this collection does not satisfy the assumptions of Theorem 0.1. But after the change of variables $\theta_1 = t_1 + t_2$, $\theta_2 = t_2$, the operators T_1 and T_2 turn into $(\partial/\partial t_2)^2$ and $(a - b)\partial/\partial t_1 + b\partial/\partial t_2$, respectively. Theorem 0.1 is applicable to this new collection if $a \neq b$ (consider the straight line passing through the points $(1, 0)$ and $(0, 2)$); thus, the space in question does not embed complementedly in a $C(K)$-space. If $a = b$, it is quite easy to realize that this space is isomorphic to a $C(K)$-space.

The second observation is that the Grothendieck theorem is not a unique reason for the absence of an isomorphism to a $C(K)$-space. This phenomenon was discussed already in [9]. We remind the reader that $\tilde{P}$ is the characteristic polynomial for a differential polynomial P, see the beginning of Section 2 for the definition. The following proposition is obvious.

Proposition 4.1. *Let E be the set of all roots of the polynomial $\tilde{P}$ that belong to $\mathbb{Z}^n$. Then the space $C^{\{P\}}(\mathbb{T}^n)$ is isomorphic to the subspace $C_E(\mathbb{T}^n)$ of $C(\mathbb{T}^n)$ that consists of all functions whose Fourier coefficients vanish on E.*

We recall that, by a commonplace in the Banach space theory (see [18] for more explanations), if the bidual of $C_E(\mathbb{T}^n)$ embeds complementedly in a $C(K)$-space, then the Fourier multiplier corresponding to the set E is bounded on $C(\mathbb{T}^n)$. By the celebrated Cohen idempotent theorem (see, e.g., the monograph [4]), this happens if and only if E belongs to the *coset ring* of the group $\mathbb{Z}^n$, i.e., to the ring of sets generated by the cosets of all subgroups of this group.

Clearly, there are differential polynomials P for which E does not belong to the coset ring of $\mathbb{Z}^n$. The simplest example of this sort (by the way, it is a mixed homogeneous polynomial) is probably

$$P_0 = 2\pi i \partial_1 - \partial_2^2, \quad n = 2. \tag{4.1}$$

Another interesting example was given in [18]. Namely, this is the operator $\partial_1^2 + \partial_2^2 - \partial_3^2$ in three variables, which is homogeneous in the usual sense. The set E for it consists of the Pythagorean triples and, surely, does not belong to the coset ring of $\mathbb{Z}^3$.

It should be noted that we do not know anything about the isomorphic type of the space generated by two operators id and P_0, where P_0 is given by (4.1).

Now, we pass to the main content of this subsection. Specifically, we want to describe a certain natural procedure of reducing the number of variables, which will lead to some new indirect applications of the nonisomorphism theorem in high dimensions. As a byproduct, it will give us an adequate language for discussing some isomorphism cases in the next subsection. An elementary step consists of reducing the number of variables by 1. Consider an index $j \in \{1, \dots, n+1\}$. A point $z = \{z_1, \dots, z_{n+1}\}$ in $\mathbb{T}^{n+1}$ will be perceived as the pair $(z^{(j)}, z_j)$, where $z^{(j)}$ is the point in $\mathbb{T}^n$ obtained from z by dropping the coordinate z_j. Let k be an integer. Consider the subspace $Y_{j,k}$ of $C^T(\mathbb{T}^{n+1})$ that consists of all functions of the form $f(z^{(j)})z_j^k$, where f is a function on $\mathbb{T}^n$.

We claim that this subspace is complemented in $C^T(\mathbb{T}^{n+1})$. Indeed, define a projection on $C(\mathbb{T}^{n+1})$ by the formula

$$(Pg)(z^{(j)}, z_j) = (2\pi)^{-1} \left(\int_{\mathbb{T}} g(z^{(j)}, \zeta)\zeta^{-k} d\zeta \right) z_j^k.$$

Since P is bounded on $C(\mathbb{T}^{n+1})$ and commutes with differentiations (the latter claim becomes clear after passage to Fourier transforms), it follows that P is bounded on $C^T(\mathbb{T}^{n+1})$, and, surely, the image of P is $Y_{j,k}$.

Now, a differential monomial $\prod_{m=1}^{n+1}\partial_m^{s_m}$, when applied to a function of the form $f(z^{(j)})z_j^k$, yields $(2\pi ik)^{s_j}\left(\prod_{m\neq j}\partial_m^{s_m}\right)f(\cdot)\times z_j^k$. It follows that $Y_{j,k}$ is isomorphic to $C^{\hat{T}}(\mathbb{T}^n)$ for a certain collection $\hat{T}$ of differential operators in n variables. This new collection will be called an *immediate successor* of T. Two cases should be distinguished. If $k=0$, then $\hat{T}$ is obtained by suppressing all differential monomials that occur in some operator in T and involve the differentiation ∂_j. This will be called an immediate successor of the first kind. If $k\neq 0$, then no such monomial is suppressed but differentiation with respect to the jth variable is replaced in it with multiplication by a nonzero constant. In this case, $\hat{T}$ is called an immediate successor of the second kind. Surely, immediate successors can be taken repeatedly with respect to different variables, which leads to “more remote” successors. (They are said to be *of the first kind* if they arise by taking a consecutive series of immediate successors of the first kind.) If a successor of the collection T gives rise to the space of smooth functions whose bidual is not complemented in a $C(K)$-space, then the space $C^T(\mathbb{T}^{n+1})$ itself has the same property.

After these preparations, we pass to specific examples. Consider the following collection A_1 of three differential operators in three variables: ∂_1, ∂_2, and $\partial_1\partial_3+\partial_2\partial_3$. It is fairly clear geometrically that there are no *admissible* affine hyperplanes in $\mathbb{R}^3$ that pass through the multiindices of two monomials involved in different operators of this collection. So, the nonisomorphism theorem is not applicable directly. However, the collection $B_1=\{\partial_1,\partial_2\}$ is an immediate successor of the first kind (in two variables) for the collection in question, and it already fits into the pattern of that theorem. Thus, $C^{A_1}(\mathbb{T}^3)$ does not embed complementedly in a $C(K)$-space.

It is easy to realize that in this situation there is also an immediate successor of the second kind (namely, $\partial_1,\partial_2,C(\partial_1+\partial_2)$) that leads to the same conclusion. This feature disappears for the collection $A_2=\{\partial_1,\partial_2,\partial_1^2\partial_3+\partial_2^2\partial_3\}$. Still, B_1 is its immediate successor of the first kind but this time no immediate successor of the second kind can be handled by the nonisomorphism theorem.

We give an instructive example of opposite nature. In $n+1$ variables ($n\geqslant 1$), consider the collection $A_3=\{\partial_1\ldots\partial_{n+1},\partial_{n+1}^2(\partial_1^n+\ldots+\partial_n^n)\}$ of two operators. We show that, for any hyperplane $\alpha_1x_1+\ldots+\alpha_{n+1}x_{n+1}=1$ with positive α_j's that passes through the point $(1,\ldots,1)$, some multiindices involved in the second operator must be situated above it. (Thus, no such hyperplane is admissible.) Indeed, otherwise we have

$$\alpha_1+\ldots+\alpha_{n+1}=1 \tag{4.2}$$

and $\alpha_jn+2\alpha_{n+1}\leqslant 1$ for $1\leqslant j\leqslant n$. Summing this group of inequalities over j, we obtain $n(\alpha_1+\ldots+\alpha_n)+2n\alpha_{n+1}\leqslant n$. Using (4.2), we arrive at $1+\alpha_{n+1}\leqslant 1$, a contradiction.

So, the nonisomorphism theorem is not applicable directly. Any immediate successor of the first kind for A_3 consists of at most one operator, and that theorem is not applicable either. However, it is applicable to an immediate successor of the second kind, namely, to the collection $B_3 = \{c_1\partial_1 \dots \partial_n, c_2(\partial_1^n + \dots + \partial_n^n)\}$ in n variables (the point $(1, \dots, 1)$ does lie on the hyperplane in $\mathbb{R}^n$ passing through the points $ne_1, \dots, ne_n$, where the e_j are the coordinate unit vectors).

It is interesting to note that if $n > 2$, then the main theorem is no longer applicable to any immediate successor of the second kind for B_3 (nor to further successors in lower dimensions, nor to successors of the first kind because the latter still consist of only one operator each): in dimension $n-1$, the point $(1, \dots, 1)$ lies *strictly below* the hyperplane passing through the points $ne_1, \dots, ne_{n-1}$. So, the procedure discussed in this subsection allows us to reduce the dimension sometimes, but not necessarily down to the value of 2. We remind the reader that 2 is the minimal dimension where nonisomorphism to a $C(K)$-space may occur. We saw nevertheless that the nature of Theorem 0.1 is two-dimensional, but this claim hides a fairly sophisticated construction (see Section 2 above).

4.2. Elliptic case

In this subsection, we state without proof a result saying that if a collection T has a dominant operator (in a sense), then the space $C^T(\mathbb{T}^n)$ is isomorphic to a $C(K)$-space. For this, we need some more notation. Let T be a finite collection of polynomials on $\mathbb{R}^n$. By its Newton polytope $\mathcal{N}(T)$ we mean the convex hull of the points $m \in \mathbb{Z}_+^n$ such that $a_m\xi^m$ is a nonzero monomial occurring in at least one polynomial $P \in T$. If the collection consists of only one polynomial P, then the polytope defined above will be referred to as the Newton polytope of P and will be denoted by $\mathcal{N}(P)$.

We assume in the remaining part of this subsection that the collections T of polynomials is such that the interior of $\mathcal{N}(T)$ is nonempty. This assumption is not essential and is imposed merely to shorten the explanations somewhat.

An $(n-1)$-dimensional face F of $\mathcal{N}(T)$ is said to be nonnegative if its outer normal has nonnegative coordinates, and it is said to be positive if this normal has positive coordinates. A face of smaller dimension is said to be nonnegative if it admits an outer normal with nonnegative coordinates. Finally, a face of smaller dimension is said to be positive if it has an outer normal with positive coordinates.

Let F be some face of $\mathcal{N}(T)$. By the F-part of a polynomial $P \in T$ we call the polynomial $\sum_{m\in F} a_m\xi^m$ if $P = \sum a_m\xi^m$.

Clearly, each positive face of dimension $n-1$ gives rise to a unique mixed homogeneity pattern (admissible hyperplane) L such that the L-senior parts of some operators in the collection T are nontrivial. (Not all admissible hyperplanes with this property arise in this way, but it is easily seen that we may disregard the others.) If Theorem 0.1 is not applicable to T, then for each positive $(n-1)$-face F of $\mathcal{N}(T)$ the linear span of the F-parts of T_j over all $j = 1, 2 \dots, J$, is one-dimensional. By using the fact that the union of all positive faces is connected, it is not hard to see that in this case $T = \{\alpha_1 P + r_1, \alpha_2 P + r_2, \dots, \alpha_l P + r_J\}$, where P is some polynomial whose monomials occur in polynomials of the collection T and correspond to multiindices that lie on positive $(n-1)$-faces of $\mathcal{N}(T)$, whereas the polynomials $r_1, r_2, \dots, r_J$ do not have monomials linked with positive faces.

The space C^T does not change if T undergoes some bijective linear transformation. Therefore, we may assume that, in the above notation, T is of the form $\{P, r_2, \dots, r_J\}$. Unfortunately, we are not able to analyze all such collections T. So, we impose two additional restrictions. The first can be viewed as a certain ellipticity condition.

Definition 4.2. We say that P is *nondegenerate* if for any nonnegative face F of $\mathcal{N}(P)$ the F-part of the corresponding characteristic polynomial $\tilde{P}$ does not have real roots except 0.

Next, we denote by $\mathcal{S}$ the *solid hull* of $\mathcal{N}(T)$, i.e., the set of all points $x \in (\mathbb{R}_+)^n$ such that $x_j \leqslant y_j$, $j = 1, \dots, n$, for some $y \in \mathcal{N}(T)$. The second assumption can be viewed as a domination condition.

Definition 4.3. We say that P is *dominant* if $\mathcal{N}(r_j) \subset \operatorname{int} \mathcal{S}$ for $j = 2, \dots, J$, where the interior is taken with respect to the topology of $(\mathbb{R}_+)^n$.

To refer to the entire setting described above, we shall say that *the initial collection T reduces to a collection with dominant and nondegenerate senior operator*. Now, we formulate the result.

Theorem 4.4. *If the collection T and all its successors of the first kind in smaller dimensions reduce to collections with a dominant and nondegenerate senior operator, then the space $C^T(\mathbb{R}^n)$ is isomorphic to a $C(K)$-space.*

We omit the fairly lengthy proof, only giving some hints. If $n = 2$, the assumption about successors is not a restriction and can be lifted. In that case, a complete proof of the theorem can be found in the preprint [13]. It can be shortened to a certain extent at the expense of a reference to somewhat similar calculations in the paper [20] (we were not aware of that paper when we wrote the preprint). In the higher dimensions, the additional assumption about successors is required indeed. A similar condition first appeared as early as in [11] for "pure anisotropic" spaces. The reasons for imposing this condition and the methods to work with it are the same in the present paper and in [11]. If we take this into account, the proof becomes quite similar to the case of dimension 2.

Finally, we mention a problem posed in the preprint [13]. There we asked about the isomorphic type of the space generated by the pair of operators $\{\mathrm{id}, \partial_1 - \sqrt{2}\partial_2\}$. Taken alone, the second operator (call it R) generates a space isomorphic to $C(\mathbb{T}^2)$ by Proposition 4.1. However, neither Theorem 0.1, nor Theorem 4.4 is applicable to the space $C^{\{\mathrm{id},R\}}(\mathbb{T}^2)$. The reason is that R does not satisfy the ellipticity condition because its characteristic polynomial vanishes somewhere on $\mathbb{R}^2$.

It has turned out that this space does embed complementedly in a $C(K)$-space. Moreover, this embedding is the operator $f \mapsto (f, Rf)$ with the image in $C(\mathbb{T}^2) \oplus C(\mathbb{T}^2)$. For a projection we can simply take the orthogonal projection from $L^2(\mathbb{T}^2) \oplus L^2(\mathbb{T}^2)$ onto the closure of this image in the L^2-metric: it can be shown that this projection is also bounded in the sup-norm. We omit the details, only signalizing that the argument applies to a fairly restricted class of examples.

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