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ADMISSIBILITY CRITERIA FOR MODEL SUBSPACES WITH FAST GROWTH OF THE ARGUMENT OF THE GENERATING INNER FUNCTION

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Let Θ be an inner function in the upper half-plane and let $K_\Theta = H^2 \ominus \Theta H^2$ be the associated model subspace of the Hardy space H^2. We call a nonnegative function ω Θ-admissible if in the space K_Θ there exists a nonzero function $f \in K_\Theta$ such that $|f| \le \omega$ a.e. on $\mathbb{R}$. We give some sufficient conditions of Θ-admissibility for the case where Θ is a meromorphic function and $\arg\Theta$ grows fast ($(\arg\Theta)'$ tends to infinity). Bibliography: 9 titles.

§1. Introduction

Let Θ be an inner function in the upper half-plane $\mathbb{C}^+$. We denote by K_Θ the space $H^2 \ominus \Theta H^2$, where H^2 is the Hardy space in $\mathbb{C}^+$ (mostly, we treat elements of this space as square summable functions on $\mathbb{R}$ with positive spectrum, or, what is the same, as boundary traces for analytic in $\mathbb{C}^+$ functions from the Hardy class $H^2(\mathbb{C}^+)$). The space K_Θ is often called *model*; this term reflects the role which is played by its vector analogs in the Nagy–Foiaş theory. The function Θ is called the *generating function of the space* K_Θ. A function $\omega = e^{-\Omega}$, where Ω is a nonnegative function defined on $\mathbb{R}$, is called an *admissible majorant for* K_Θ (or a Θ-*admissible* function) if there exists a nonzero function $f \in K_\Theta$ such that

$$|f| \le \omega \text{ a.e. on } \mathbb{R}.$$

Thus, the fact that a given nonnegative function is not Θ-admissible is equivalent to the following uniqueness theorem:

$$f \in K_\Theta, \quad |f| \le \omega \text{ a.e. on } \mathbb{R} \Rightarrow f = 0.$$

A dual approximative reformulation of the above property can be found in [3].

Admissibility of majorants for model spaces was studied in the papers [6, 7, 1, 3, 4, 9]. In particular, the last paper contains a new proof of the Beurling–Malliavin theorem on multiplicator which uses neither the apparatus of complex analysis nor potential theory. The purpose of the present paper is to find sufficient conditions of admissibility for several particular spaces K_Θ, where Θ is a meromorphic Blaschke product B (in the half-space $\mathbb{C}^+$).

The same problem was studied in the paper [4]. Mostly, the spaces studied in [4] are different from the spaces which we study. Our methods are different from methods of [4] as well.

In the paper [6], quite exact criteria of admissibility of majorants were found for products B with "vertical" (i.e., purely imaginary) zeros; these results were refined and extended by A. Baranov in [1]. Results of the above-mentioned papers indicate that criteria of B-admissibility are essentially different depending on the behavior of the argument of the Blaschke product B on $\mathbb{R}$. Recall that if a function Θ is meromorphic and interior (in $\mathbb{C}^+$), then this function is analytic on the real line $\mathbb{R}$ and can be represented as

$$\Theta = e^{i\varphi},$$

where φ is a real-analytic function which is defined on $\mathbb{R}$ up to a real additive constant that is a multiple of 2π; we call the function ϕ the *argument of the function* Θ and denote it $\arg\Theta$.

A meromorphic inner function has the form

$$\Theta(z) = e^{iaz} \prod_{k\in\mathbb{Z}} \frac{z - z_k}{z - \overline{z}_k} \cdot e^{i\alpha_k}, \quad \Im z > 0,$$

where $a \ge 0$ and the sequence $\{z_k\}$ is such that $\Im z_k > 0$, $\lim\limits_{k\to\infty} |z_k| = \infty$, $z_k = x_k + iy_k$, $x_k \in \mathbb{R}$, $y_k > 0$, and

$$\sum_{k\in\mathbb{Z}} \frac{y_k}{|z_k|^2} < +\infty \quad \text{(the Blaschke condition);}$$

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the unimodular factors $e^{i\alpha_k}$ provide convergence of the product. The function $\arg\Theta$ is increasing, and

$$\frac{d}{dx}\arg\Theta(x) = a + \sum_{k\in\mathbb{Z}} \frac{2y_k}{(x-x_k)^2 + y_k^2}, \quad x \in \mathbb{R} \tag{1.1}$$

(see [6, p. 1259]).

It was shown in the papers [6, 7, 1] that criteria of Θ-admissibility essentially depend on the velocity of rotation of the unit vector $\Theta(x)$ around the origin as the real variable x grows; more exactly, they depend on the growth of the function $\arg\Theta$. Roughly speaking, it is easier to analyze the Θ-admissibility of a given majorant for slow rotation of the unit vector $\Theta(x)$; such an analysis is the more problematic the faster is the rotation. The vector $B(x)$, where B is a meromorphic Blaschke product, rotates the faster the closer to the real line $\mathbb{R}$ are the roots z_k. Thus, in the extremal case where B is a *finite* product, the curve $y = \arg B(x)$ is quite flat, and criteria of admissibility for majorants are obvious (in this case, K_B is precisely the class of rational fractions $\frac{P(z)}{(z-z_1)\cdots(z-z_N)}$, where $\deg(P) < N$).

In the case where roots of the product B are either "vertical" or "almost vertical" (i.e., in a sense, they are far from $\mathbb{R}$), the typical growth of the arguments is still slow, and its rate decays ($(\arg B)'(x) = o(1)$, $|x| \to \infty$). In this case, B-admissibility can be analyzed almost completely. "Tentatively," in the paper [7] several sufficient conditions of B-admissibility for a majorant ω were obtained for "horizontal roots" of the form $z_k = \operatorname{sgn}(k)|k|^\alpha + i$, $\alpha \in (1/2, 1)$, $k \in \mathbb{Z}$. (The restriction $\alpha > 1/2$ is imposed by the Blaschke condition, while the case $\alpha \geq 1$ corresponds to a relatively slow growth of the function $\arg B$ and was analyzed in [6]). In our paper, these particular results of "experimental" character are included into a more general scheme which covers sequences of roots of the form

$$z_k = f(k) + i, \quad k \in \mathbb{Z}, \tag{1.2}$$

and, further, of the form

$$z_k = f(k) + ig(k), \quad k \in \mathbb{Z}, \tag{1.3}$$

where f is a real-valued increasing function and g is a positive function of a real variable; these functions are subjected to some natural conditions (including, of course, the Blaschke condition). Our assumptions cover not only "purely horizontal" cases ($g = \text{const}$) or close cases of roots belonging to a band ($\inf_{s\in\mathbb{R}} g(s) > 0$ and $\sup_{s\in\mathbb{R}} g(s) < +\infty$) but also some cases of unbounded growth of the ordinates $g(k)$ ($\lim_{|x|\to\infty} g(x) = +\infty$). The mostly problematic case of zeros that approach the axis $\mathbb{R}$ ($\lim_{|x|\to\infty} g(x) = 0$) is considered in this paper as well.

Compared to [7], we advance not only in generality of the cases considered but also in the character of admissibility conditions.

In [7], admissibility conditions are imposed not on the majorant ω itself (or, what is the same, on Ω) but on the Hilbert transform $\widetilde{\Omega}$ of the function Ω. We apply a recent F. L. Nazarov's result on "correction" of the Hilbert transfrom of a Lipschitz continuous function (see [9]). This approach allows us to find sufficient admissibility conditions which are imposed on Ω itself; it is relatively easy to check such conditions.

This paper is devoted to sufficient admissibility conditions only. The problem of their exactness is open. Some results in the latter direction can be found in [2].

We use the following notation: $\mathbf{P}$ denotes the Poisson measure on $\mathbb{R}$:

$$d\,\mathbf{P}(x) = \frac{1}{\pi}\cdot\frac{dx}{1+x^2};$$

$\mathcal{L}(F)$ is the logarithmic integral of a nonnegative function F defined on $\mathbb{R}$:

$$\mathcal{L}(F) = \int_{\mathbb{R}} \log F\, d\,\mathbf{P};$$

$\mathrm{Lip}_\alpha(I)$ denotes the space of functions that satisfy the Lipschitz condition of order α on an interval I:

$$f \in \mathrm{Lip}_\alpha(I) \Leftrightarrow |f(x) - f(y)| \leq C_f|x-y|^\alpha, \quad x, y \in I.$$

In our study of admissibility of a majorant, we assume that the majorant is strictly positive and bounded. Recall that the finiteness of the integral $\mathcal{L}(\omega)$ (i.e., the inequality $\mathcal{L}(\omega) > -\infty$) is *necessary* for Θ-admissibility.

We also note that if, for a fiven majorant, we can find a Θ-admissible majorant ω_1 such that $\omega \geq \omega_1$, then the majorant ω is Θ-admissible as well. Thus, working with positive continuous majorants ω, we may assume that they are arbitrarily smooth (indeed, for any positive function $\omega \in C(\mathbb{R})$ with finite integral $\mathcal{L}(\omega)$ it is easy to construct a positive function $\omega_1 \in C^\infty(\mathbb{R})$ such that $\mathcal{L}(\omega_1) > -\infty$ and $\omega_1 \leq \omega$).

We permanently use the Hilbert transform, which we treat as an operator that takes any function $F \in L^1(\mathbf{P})$ to a function $\widetilde{F}$,

$$\widetilde{F}(x) = \frac{1}{\pi}(p.v.)\int\limits_{\mathbb{R}} F(t)\left(\frac{1}{x-t} + \frac{t}{t^2+1}\right)dt; \tag{1.4}$$

it is known that this integral converges almost everywhere on $\mathbb{R}$.

The structure of the paper is as follows. In Sec. 2, we first restrict our consideration to a particular case ($z_k = \operatorname{sgn}(k)|k|^\alpha + i$, $\alpha \in (1/2, 1)$), which was treated in [7], and get rid of a bulky condition of B-admissibility used in [7, Theorem 1.10]; this condition is replaced by a simple condition on Ω (instead og $\widetilde{\Omega}$) in Theorem 3 of Sec. 2.

In Theorem 4 (Sec. 2.2), we find an asymptotic of the function $(\arg B)'$ for a relatively general class of Blaschke products with "purely horizontal" zeros. This asymptotic is applied in Theorem 5 to get sufficient conditions of B-admissibility for such products B; these conditions are formulated with application of the function $\widetilde{\Omega}$. Theorem 5 treats the case of "purely horizontal" zeros.

In Sec. 3, we apply techniques of the paper [9] to get rid of the Hilbert transform in sufficient conditions of B-admissibility. For this purpose, we prove a "correcting theorem" for the Hilbert transform (similarly to [9]); in our opinion, this result is of independent interest (see Theorem 7).

In Sec. 4, we turn to products B that correspond to zeros $z_k = f(k)+ig(k)$, $k \in \mathbb{Z}$, where the function f grows from $-\infty$ to $+\infty$, and the function g is positive and may grow to $+\infty$. Theorem 9 proved in Sec. 4 is illustrated by Corollary 9.1 in which we give simple sufficient conditions of B-admissibility for products B constructed by zeros $z_k = \operatorname{sgn}(k)|k|^\alpha + i|k|^\gamma$, $k \in \mathbb{Z} \setminus \{0\}$, where $1/2 < \alpha < 1$ and $0 \leq \gamma < 2\alpha - 1$. Note that our main result, Theorem 9, formally contains the "purely horizontal" Theorem 8 (we prefer to give a separate proof of the latter theorem, which is free of complications related to unbounded growth of ordinates of zeros). Comparison of these theorems indicates a definite stability of our admissibility conditions: in the case of growth of ordinates which we allow, admissibility conditions remain the same as for purely horizontal zeros.

The final Sec. 5 reflects our (not very large) knowledge concerning the case of zeros that unboundedly approach the real line $\mathbb{R}$.

§2. Admissibility conditions in terms of the function $\widetilde{\Omega}$

Following [7], we say that a function $f : \mathbb{R} \to \mathbb{R}$ is *mainly increasing* is the following conditions are satisied:

(1) the function f is absolutely continuous;

(2a) there exists an increasing sequence $\{d_n\}_{n\in\mathbb{Z}}$ such that $f(d_n) = 2\pi n$, $\lim\limits_{n\to-\infty} d_n = -\infty$, and $\lim\limits_{n\to+\infty} d_n = +\infty$;

(2b) the sequence $l_n = \frac{1}{2}(d_n - d_{n-1})$ is bounded;

(3) there exists a constant C such that

$$\frac{1}{2l_n}\int\limits_{d_{n-1}}^{d_n} |f'(x) - f'(t)|dt \leq C \tag{2.1}$$

for any $x \in (d_{n-1}, d_n)$ and $n \in \mathbb{Z}$.

The following statement is the main source of admissibility conditions in the paper [7].

Theorem 1 (Havin–Mashreghi)**.** *If the function $\Omega = -\log\omega$ is bounded from below and the function $\arg\Theta + 2\widetilde{\Omega}$ is mainly increasing, then the function ω is an admissible majorant for the space K_Θ.*

It is shown in the paper [3] that Theorem 1 remains valid if condition (2b) is omitted.

In this paper, we consider the following Blaschke product as the function Θ:

$$\Theta(z) = B(z) = \prod_{k=-\infty}^{k=+\infty} \frac{1 - \frac{z}{z_k}}{1 - \frac{z}{\overline{z}_k}}, \qquad \lim_{|k|\to+\infty} |z_k| = +\infty; \tag{2.2}$$

in this case, the argument $\arg\Theta(x)$ is fast growing. We assume that $\sum\limits_{k\in\mathbb{Z}} \frac{\Im z_k}{|z_k|^2} < +\infty$ (the condition of convergence for a Blaschke product). These conditions are satisfied if $z_k = \operatorname{sgn}(k)|k|^\alpha + i$, $\frac{1}{2} < \alpha$; the corresponding Blachke product is denoted B_α. We consider the case

$$\alpha \in \big(1/2, 1\big). \tag{2.3}$$

The case $\alpha \geq 1$ has been analyzed in the papers [6, 7] (from a different point of view). Under condition (2.3), the following sufficient admissibility conditions were obtained in [7].

Theorem 2 (Havin–Mashreghi). *Assume that a function* Ω *has the following properties:* $\int\limits_{\mathbb{R}} \Omega d\,\mathbf{P} < +\infty$, $\widetilde{\Omega} \in C^1(\mathbb{R})$, *and* $\Omega \geq 0$. *Assume, in addition, that*

(a) $$-\frac{\pi}{\alpha} \leq \lim_{|x|\to\infty} \inf \frac{\widetilde{\Omega}'(x)}{|x|^{\frac{1}{\alpha}-1}} \leq \lim_{|x|\to\infty} \sup \frac{\widetilde{\Omega}'(x)}{|x|^{\frac{1}{\alpha}-1}} < C \quad \text{for some} \quad C > 0;$$

(b) $\lambda_t(t^{1-\frac{1}{\alpha}}) \leq K$, *where* λ_t *is the continuity modulus of the function* $\widetilde{\Omega}'$ *on* $\mathbb{R}\backslash(-t,t)$, *i.e.*,

$$\lambda_t(\delta) = \sup\big\{|\widetilde{\Omega}'(t_1) - \widetilde{\Omega}'(t_2)| : |t_1|, |t_2| \geq t, |t_1 - t_2| < \delta\big\}.$$

Then the function $\omega = e^{-\Omega}$ *is admissible for the space* K_{B_α}.

2.1. Preliminary remarks. In what follows, we need two simple statements. Denote by S the Steklov averaging operator: $Sf(x) = \frac{1}{2}\int\limits_{x-1}^{x+1} f(s)ds$.

Proposition 1. *If a function* f *is differentiable, then*

$$|Sf(x) - f(x)| \leq \sup_{|s-x|<1} |f'(s)| \quad \text{and} \quad (Sf)'(x) = \frac{1}{2}(f(x+1) - f(x-1)) = f'(s),$$

where $s = s(x) \in [x-1, x+1]$.

Proposition 2. *Let* $\beta \in (0,1)$ *and let* $\Omega(x) = |x|^\beta$, $x \in \mathbb{R}$. *Then the function* $\widetilde{\Omega}$ *satisfies conditions* (a) *and* (b) *of Theorem* 2 *(thus, the majorant* ω, *where* $\omega(x) = e^{-|x|^\beta}$, *is admissible for the space* K_{B_α}*).*

In Proposition 2, the function Ω is summable with respect to the Poisson measure for any $\beta \in (0,1)$ and extremely regular; at the same time, for many "vertical" (purely imaginary) sequences of zeros, K_B-admissibility takes places not for any majorant $e^{-|x|^\beta}$, $0 < \beta < 1$ (see examples in [7]).

Proof. It is known that

$$(\widetilde{\Omega})'(x) = c_\beta((\operatorname{sgn} x)|x|^\beta)' = c_\beta \operatorname{sgn}(x)\beta|x|^{\beta-1}$$

for any $x \in \mathbb{R}\backslash\{0\}$; thus, the function $\widetilde{\Omega}'$ satisfies condition (a) of Theorem 2 since $\beta - 1 < 0 < \frac{1}{\alpha} - 1$. On the other hand, the estimate $\widetilde{\Omega}''(x) = O(|x|^{\frac{1}{\alpha}-1})$ implies condition (b) (see [7, p. 1300]). Since $\widetilde{\Omega}''(x) = c_\beta \operatorname{sgn}(x)|x|^{\beta-2}$ ($x \in \mathbb{R}\backslash\{0\}$), condition (b) of Theorem 2 is satisfied as well. $\square$

Theorem 3. *In Theorem* 2, *condition* (b) *can be replaced by the following two conditions:*

(1) $\widetilde{\Omega} \in C^2(\mathbb{R})$;

(2) $\Omega'(x) = O(|x|^\beta)$, $|x| \to \infty$, *for some* $\beta < 1$ *(let us emphasize that* β *is not related to* α*).*

Condition 1 is not essential (see the Introduction).

Proof. 1. It was noted that condition (b) of Theorem 2 is satisfied if $\widetilde{\Omega}''(x) = O(|x|^{\frac{1}{\alpha}-1})$ (see [7, p. 1300]).

2. The above estimate of the second derivative is valid for the function $S\Omega$: $|\widetilde{S\Omega}''(x)| \leq 2\sup\limits_{|s-x|\leq 1} |\widetilde{\Omega}'(s)| = O(|x|^{\frac{1}{\alpha}-1})$ since $(\widetilde{S\Omega})' = S[(\widetilde{\Omega})]'$ (the operators S and $\widetilde{\ }$ commute, see [7, pp. 1286–1287] for details). Further, $|S\Omega(x) - \Omega(x)| = |\Omega(x_1) - \Omega(x)| \leq |\Omega'(x_2)| = O(|x|^\beta)$ (see Proposition 1) for some x_1 and x_2 from the interval

$[x-1,x+1]$. Hence, the function $S\Omega$ is summable with respect to the Poisson measure, and there exist constants $C_{\Omega,1}$ and $C_{\Omega,2}$ such that

$$S\Omega(x) + C_{\Omega,1}|x|^{\beta} + C_{\Omega,2} \geq \Omega(x) \tag{2.4}$$

for any $x \in \mathbb{R}$. On the other hand, the function $S\Omega(x) + C_{\Omega,1}|x|^{\beta} + C_{\Omega,2}$ satisfies conditions (a) and (b) of Theorem 2 since $\widetilde{|x|^{\beta}}' = c_{\beta}\operatorname{sgn}(x)\beta|x|^{\beta-1} \overset{|x|\to\infty}{\longrightarrow} 0$ and $\widetilde{|x|^{\beta}}'' = c_{\beta}\operatorname{sgn}(x)\beta(\beta-1)|x|^{\beta-2} \overset{|x|\to\infty}{\longrightarrow} 0$. Hence, the function ω defined by the equality

$$\omega(x) = e^{-(S\Omega(x)+C_{\Omega,1}|x|^{\beta}+C_{\Omega,2})}, \quad x \in \mathbb{R},$$

is admissible. We see that the function $\omega_1 = e^{-\Omega}$ is admissible as well since $\omega_1 \geq \omega$ (see inequality (2.4)). $\square$

2.2. Estimation of the derivative of the argument of a Blaschke product. Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function such that $\int_{\mathbb{R}} \frac{dt}{1+f^2(t)} < +\infty$. Denote by B_f the Blaschke product with zeros $z_k = f(k) + i$, $k \in \mathbb{Z}$. Since we are interested in values of f at integer points, we may assume that the function f is differentiable. The convergence of the integral $\int_{\mathbb{R}} \frac{dt}{1+f^2(t)}$ guarantees the convergence of the Blaschke product. It is easily seen that $\lim_{t\to\pm\infty} f(t) = \pm\infty$. By (1.1),

$$\frac{d}{dx}\arg B_f(x) = 2\sum_{k=-\infty}^{+\infty} \frac{1}{(x-f(k))^2+1} = 2\sigma(x). \tag{2.5}$$

Take $x > f(0)$; let l be a number such that $f(l) \leq x < f(l+1)$. Then $l = f^{-1}(x) + O(1)$. Decompose the sum into three parts:

$$\sigma(x) = \Big(\sum_{k=-\infty}^{0} + \sum_{k=1}^{l} + \sum_{k=l+1}^{+\infty}\Big)\frac{1}{(x-f(k))^2+1} = \Sigma_1 + \Sigma_2 + \Sigma_3.$$

Note that

$$\Sigma_1 = \sum_{k=0}^{+\infty} \frac{1}{(x-f(-k))^2+1} \leq \sum_{k=1}^{+\infty} \frac{1}{(f(0)-f(-k))^2+1} < +\infty.$$

To estimate the sums Σ_2 and Σ_3, set

$$F(t) = \frac{1}{(x-f(t))^2+1}, \quad t > 0.$$

Then

$$\Sigma_2 = \sum_{k=1}^{l} F(k) \quad \text{and} \quad \Sigma_3 = \sum_{k=l+1}^{+\infty} F(k).$$

Note that the function F is increasing in the interval $[0, f^{-1}(x)]$ and decreasing in the interval $[f^{-1}(x), +\infty)$. Thus,

$$F(1) + \int_1^l F(t)dt \leq \Sigma_2 \leq \int_1^l F(t)dt + 1 \quad (0 \leq F \leq 1),$$

i.e.,

$$\Sigma_2 = \int_1^l F(t)dt + O(1) = \int_0^{f^{-1}(x)} F(t)dt + O(1),$$

$$F(l+1) + \int_{l+1}^{+\infty} F(t)dt \geq \Sigma_3 \geq \int_{l+1}^{+\infty} F(t)dt, \quad \Sigma_3 = \int_{f^{-1}(x)}^{+\infty} F(t)dt + O(1).$$

Set $h(x) = (f^{-1})'(x)$. Assume that $h \in \mathrm{Lip}_\beta(\mathbb{R})$ for some $\beta \in (0,1)$. Then

$$\int\limits_0^{f^{-1}(x)} F(t)dt = \int\limits_0^{f^{-1}(x)} \frac{dt}{(x-f(t))^2+1} = \int\limits_0^x \frac{h(x-z)}{z^2+1}dz = \int\limits_0^x \frac{h(x)}{z^2+1}dz + \int\limits_0^x \frac{h(x-z)-h(x)}{z^2+1}dz$$

$$\le h(x)\int\limits_0^x \frac{dz}{z^2+1} + C_f\int\limits_0^x \frac{z^\alpha}{z^2+1}dz \le \frac{\pi}{2}h(x) + C_2$$

for $x > 0$. On the other hand,

$$\int\limits_0^x \frac{h(x)}{z^2+1}dz = h(x)\frac{\pi}{2} - h(x)\int\limits_x^{+\infty} \frac{dz}{z^2+1} \ge \frac{\pi}{2}h(x) - \frac{h(x)}{x} \ge \frac{\pi}{2}h(x) - C_4.$$

A similar estimate holds for the integral $\int\limits_{f^{-1}(x)}^{+\infty} F(t)dt$.

Thus, we have proved the following statement.

Theorem 4. *Assume that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions (where $h = h(f) = (f^{-1})'$):*

(a) *the function f is strictly increasing;*

(b) *$h \in \mathrm{Lip}_\beta(\mathbb{R})$ for some $\beta \in (0,1)$;*

(c) $\displaystyle\int\limits_{\mathbb{R}} \frac{dt}{f^2(t)+1} < +\infty.$

Then the Blaschke product with zeros $z_k = f(k) + i$, $k \in \mathbb{Z}$, satisfies the following relation:

$$\frac{d}{dx}\arg B_f(x) = 2\pi h(x) + O(1), \quad |x| \to \infty. \tag{2.6}$$

Remark 1. If f is a differentiable, strictly increasing function such that $f(x) = \mathrm{sgn}\,(x)|x|^\alpha \log^\delta(|x|+2)$, $|x| \ge C_\delta$, $(\frac{1}{2} < \alpha < 1, \delta \in \mathbb{R})$, then f satisfies the conditions of Theorem 4.

Remark 2. Let B_1 and B_2 be Blaschke products constructed by zeros $\{z_{k,1}\}_{k\in\mathbb{Z}}$ and $\{z_{k,2}\}_{k\in\mathbb{Z}}$, respectively, and such that

$$\text{(a)} \quad \Re(z_{k,1}) = \Re(z_{k,2}); \qquad \text{(b)} \quad 0 < \inf_{k\in\mathbb{Z}} \frac{\Im(z_{k,1})}{\Im(z_{k,2})} \le \sup_{k\in\mathbb{Z}} \frac{\Im(z_{k,1})}{\Im(z_{k,2})} < \infty.$$

Then there exist positive constants c and C such that

$$c \le \frac{\frac{d}{dx}\arg B_1(x)}{\frac{d}{dx}\arg B_2(x)} \le C,$$

for any $x \in \mathbb{R}$.

Proof. Let $z_{k,j} = x_k + iy_{k,j}$, $j = 1,2$. If $y_1^2 \le y_2^2$, then $\frac{a^2+y_1^2}{a^2+y_2^2} \le 1$ for any $a \in \mathbb{R}$; if $y_1^2 > y_2^2$, then

$$\frac{a^2+y_1^2}{a^2+y_2^2} = 1 + \frac{y_1^2-y_2^2}{a^2+y_2^2} \le \frac{y_1^2}{y_2^2}$$

for any $a \in \mathbb{R}$. Hence, if $y_1/y_2 \le C$, then

$$\frac{y_2}{a^2+y_2^2}\Big/\frac{y_1}{a^2+y_1^2} \le \max(1, y_1/y_2) \le \max(1, C)$$

for any $a \in \mathbb{R}$. It follows that if $y_{k,1}/y_{k,2} \le C$, $k \in \mathbb{Z}$, then

$$\frac{d}{dx}\arg B_1(x) = \sum_{k=-\infty}^{k=+\infty} \frac{2y_{k,1}}{(x_k-x)^2+y_{k,1}^2} \le \max(1,C)\sum_{k=-\infty}^{k=+\infty} \frac{2y_{k,2}}{(x_k-x)^2+y_{k,2}^2} = \max(1,C)\frac{d}{dx}\arg B_2(x)$$

for any $x \in \mathbb{R}$. $\square$

Remark 3. For Blaschke products constructed by zeros $z_k = f(k) + ig(k)$, where $0 < c \le g(k) \le C$, we get the estimate

$$(\arg B)'(x) \asymp h(x)$$

instead of the asymptotic formula of Theorem 4.

2.3. Admissibility conditions for the space K_{B_f}. Now we are ready to prove some sufficient admissibility conditions for the space K_{B_f}.

Theorem 5. *Assume that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:*

(a) *f is strictly increasing and differentiable;*

(b) *$h(= h_f) \in \mathrm{Lip}_\alpha(\mathbb{R})$, $0 < \alpha < 1$;*

(c) $\displaystyle\int_{\mathbb{R}} \frac{dt}{1+f^2(t)} < +\infty.$

(These are the conditions of Theorem 4.) Assume, in addition, that $\lim\limits_{x\to\infty} f'(x) = 0$. Then any smooth function $\omega = e^{-\Omega}$ such that

(1) $\displaystyle\int_{\mathbb{R}} \Omega d\mathbf{P} < +\infty$*;*

(2) *$\Omega'(x) = O(|x|^\beta)$ for some $\beta \in (0,1)$;*

(3) *$|\widetilde{\Omega}'(x)| \le Ah(x)$ for some $A < 2\pi$ and any $x \in \mathbb{R}$,*

is admissible for the space K_{B_f}. (Recall that B_f is the Blaschke product with zeros $z_k = f(k) + i$, $k \in \mathbb{Z}$.)

This theorem has the same type as Theorem 2; condition (b) of Theorem 2 is replaced by a simpler estimate (2).

Proof. By Theorem 1, it is enough to show that the function $\Phi_f = \arg B_f + 2\widetilde{\Omega}_1$ is *mainly increasing* for some majorant $\Omega_1 \ge \Omega$ for which the integral $\int_{\mathbb{R}} \Omega_1 d\mathbf{P}$ is finite. Note that

$$(2\pi - A)h \le \frac{d}{dx}\Phi_f \le (2\pi + A)h; \tag{2.7}$$

hence, $\frac{d}{dx}\Phi_f \ge c > 0$ (the function h is separated from zero since $\sup\limits_{\mathbb{R}} |f'| < +\infty$). Hence, there exists a strictly increasing sequence d_n such that $\Phi_f(d_n) = 2\pi n$, $n \in \mathbb{Z}$. It is easily seen that the sequence $l_n = d_{n+1} - d_n$ is bounded. Thus, it remains to check condition (2.1).

For the function Ω, the estimate $|\widetilde{\Omega}'| \le C_{\Omega,3}h$ is valid. Taking (if necessary) the function $S\Omega + C_1|x|^\beta + C_2$ as the majorant Ω_1 (recall that $|S\Omega(x) - \Omega(x)| \le C_1|x|^\beta + C_2$, $x \in \mathbb{R}$), we may assume that $|\widetilde{\Omega}''| \le C_{\Omega,3}h + C_{\Omega,4}$.

Set $\phi(x) = \frac{d}{dx}\arg B_f(x) - 2\pi h(x)$. The function ϕ is bounded (see Theorem 4). Further,

$$\frac{1}{2l_n}\int_{d_n}^{d_{n+1}} |\Phi_f'(x) - \Phi_f'(t)|dt \le \frac{\pi}{l_n}\int_{d_n}^{d_{n+1}} |h(x) - h(t)|dt + \frac{1}{2l_n}\int_{d_n}^{d_{n+1}} |\phi(x) - \phi(t)|dt + \frac{1}{2l_n}\int_{d_n}^{d_{n+1}} |\widetilde{\Omega}'(x) - \widetilde{\Omega}'(t)|dt$$

$$\le C\frac{\pi}{2l_n}\int_{d_n}^{d_{n+1}} |x-t|^\alpha dx + \frac{d_{n+1}-d_n}{l_n}\max_{s\in\mathbb{R}}|\phi(s)| + \frac{1}{2l_n}\int_{d_n}^{d_{n+1}} (d_{n+1}-d_n)|\widetilde{\Omega}''(\zeta(t))|dt,$$

where $\zeta(t) \in [d_n, d_{n+1}]$. The first and second terms are bounded. The estimate $|\widetilde{\Omega}''| \le C_{\Omega,3}h + C_{\Omega,4}$, Lipschitz continuity of the function h, and boundedness of the sequence l_n imply that the third term is bounded as well:

$$\frac{1}{2l_n}\int_{d_n}^{d_{n+1}} (d_{n+1}-d_n)|\widetilde{\Omega}''(\zeta(t))|dt \le C_0 + C_{\Omega,3}\int_{d_n}^{d_{n+1}} h(\zeta(t))dt = C_0 + C_{\Omega,3}\Bigg(\int_{d_n}^{d_{n+1}} h(t)dt + \int_{d_n}^{d_{n+1}} [h(\zeta(t)) - h(t)]dt\Bigg)$$

$$\le C_1 + C_2\int_{d_n}^{d_{n+1}} h(t)dt \le C_1 + C_2(f^{-1}(d_{n+1}) - f^{-1}(d_n)) = C_1 + C_2(f^{-1}(\Phi_f^{-1}(2\pi n + 2\pi)) - f^{-1}(\Phi_f^{-1}(2\pi n)))$$

$$= C_1 + C_2(2\pi(f^{-1}\circ\Phi_f^{-1})'(\xi)) = C_1 + C_2 2\pi\frac{h(\Phi_f^{-1}(\xi))}{\Phi_f'(\Phi_f^{-1}(\xi))} \le C_3,$$

where $2\pi n \le \xi \le 2\pi n + 2\pi$. □

Renark 4. If f is a differentiable, strictly increasing function such that $f(x) = \operatorname{sgn}(x)|x|^\alpha \log^\delta(|x|+2)$ for $|x| \ge 1$ ($\frac{1}{2} < \alpha < 1$), then f satisfies the conditions of Theorem 5 for any $\delta \in \mathbb{R}$.

§3. Admissibility conditions in terms of the function Ω

Admissibility conditions of Theorem 5 include, in addition to the function Ω, the function $\widetilde{\Omega}$. This is the main disadvantage of Theorem 5. It is not necessary to prove the admissibility of the function Ω. It is enough to establish the admissibility of some function Ω_1 such that $\Omega_1 \geq \Omega$. Thus, the problem of existence of a majorant Ω_1 of the function Ω with necessary properties is of considerable interest.

Recently, a theorem on existence of such a majorant with a Lipschitz continuous Hilbert transform has been proved in the paper [9].

Theorem 6 (F. L. Nazarov)**.** *Let Ω be a nonnegative function from the class $L^1(\mathbf{P})\bigcap \mathrm{Lip}_1(\mathbb{R})$. For any $\varepsilon > 0$ there exists a function Ω_1 such that*

$$\text{(a)}\quad \Omega_1 \geq \Omega; \qquad \text{(b)}\quad \Omega_1 \in \mathrm{Lip}_1(\varepsilon, \mathbb{R});$$
$$\text{(c)}\quad \Omega_1 \in L^1(\mathbf{P}); \qquad \text{(d)}\quad \widetilde{\Omega_1} \in \mathrm{Lip}\,(\varepsilon, \mathbb{R}).$$

The proof of Theorem 6 is based on the existence of a local majorant (established by Nazarov, see [9]).

Let us denote by cI the interval of length $c|I|$ having the same center as an interval I.

Basic Lemma. *Let I be a closed bounded interval. Consider a function $f \in \mathrm{Lip}_1(K, I)$ such that $f \geq 0$ and $\|f\|_\infty \leq \delta|I|$, where δ is a positive number and $K \geq 1$. There exists a function $F \in C^\infty(\mathbb{R})$, $F \geq 0$, such that*

(1) $F \equiv 0$ *outside* $\frac{3}{2}I$;

(2) $F \geq f$ *on* I;

(3) $\|F'\|_\infty \leq A\delta$;

(4) $\|\widetilde{F}'\|_\infty \leq A\delta$;

(5) $\int\limits_I f(s)ds \geq \frac{A\delta}{K}\int\limits_{\mathbb{R}} F(s)ds$,

where A is an absolute constant.

We need the following estimate of growth for the function Ω.

Proposition 3. *Assume that a function λ monotonically increases on the ray $[0, +\infty)$ and monotonically decreases on the ray $(-\infty, 0]$. Let Ω be a positive differentiable function such that $\Omega \in L^1(\mathbf{P})$, $|\Omega'| \leq \lambda$, and $\Omega(0) = 0$. Then $\Omega(x) = o(|x|\sqrt{\lambda(x)})$ (both as $x \to +\infty$ and as $x \to -\infty$).*

Proof. Without loss of generality, we may take $x > 0$. Let us estimate the function Ω: $\Omega(x) = \int\limits_0^x \Omega'(t)dt \leq \int\limits_0^x \lambda(t)dt \leq x\lambda(x)$. Hence, if $0 \leq x - \frac{\Omega(x)}{2\lambda(x)} \leq y \leq x$, then

$$\Omega(y) = \Omega(x) + \Omega(y) - \Omega(x) = \Omega(x) + \Omega'(\xi)(y-x) \geq \Omega(x) + \lambda(\xi)(y-x) \geq \Omega(x) + \lambda(x)(y-x).$$

Further,

$$\int\limits_{x-\frac{\Omega(x)}{2\lambda(x)}}^{x} \frac{\Omega(y)}{1+y^2}dy \geq \frac{1}{1+x^2}\int\limits_{x-\frac{\Omega(x)}{2\lambda(x)}}^{x} \Omega(y)dy \geq \frac{1}{1+x^2}\int\limits_{x-\frac{\Omega(x)}{2\lambda(x)}}^{x} (\Omega(x) + \lambda(x)(y-x))dy = \frac{3}{8}\cdot\frac{\Omega^2(x)}{(1+x^2)\lambda(x)}.$$

Hence,

$$\frac{\Omega(x)}{\sqrt{1+x^2}\sqrt{\lambda(x)}} \leq 3\Bigg(\int\limits_{x-\frac{\Omega(x)}{2\lambda(x)}}^{+\infty} \frac{\Omega(y)}{1+y^2}dy\Bigg)^{\frac{1}{2}}.\square$$

Theorem 7. *Let Ω be a twice differentiable positive function such that $\Omega \in L^1(\mathbf{P})$, $|\Omega'| \leq \lambda$, and $\Omega(0) = 0$, where the function λ has the following properties:*

(1) $\lambda \geq 1$;

(2) λ *increases monotonically on the ray* $[0, +\infty)$ *and decreases monotonically on the ray* $(-\infty, 0]$;

(3) $\lim\limits_{|x|\to+\infty} \lambda(x) = +\infty$;

(4) $\sup\limits_{x\in\mathbb{R}} \left| \frac{\lambda(x)}{\lambda(\frac{x}{2})} \right| \leq +\infty$.

Then for any $\varepsilon > 0$ there exists a function Ω_1 such that

$$\text{(a)} \quad \Omega_1 \geq \Omega; \qquad \text{(b)} \quad \Omega_1 \in L^1(\mathbf{P});$$
$$\text{(c)} \quad |\widetilde{\Omega}_1'(x)| \leq \varepsilon\lambda(x); \qquad \text{(d)} \quad |\Omega_1'(x)| \leq \varepsilon\lambda(x).$$

Proof. Decompose the real line $\mathbb{R}$ into intervals as follows: $\mathbb{R} = I_0 \cup \bigcup\limits_{k=N}^{\infty} I_k \cup \bigcup\limits_{k=-N}^{-\infty} I_k$, where

$$I_0 = [-2^N, 2^N], \quad I_k = [2^k, 2^{k+1}], \quad k \geq N,$$

and

$$I_k = [-2^{k+1}, -2^k], \quad k \leq -N.$$

We fix the number N later. Denote $f_j = \Omega\,|_{I_j}$, $j > 0$. If $x > 0$, then

$$|f_j(x)| \leq A_1 x \sqrt{\lambda(x)} \leq 2A_1 2^j \sqrt{\lambda(2^{j+1})}$$

and

$$|f_j'(x)| \leq \lambda(x) \leq \lambda(2^{j+1}), \quad x \in I_j.$$

Set $g_j(x) = \frac{f_j(x)}{\lambda(2^{j+1})}$, $x \in I_j$. Fix $\delta > 0$. If a number $N = N(\delta)$ is large enough, then

$$|g_j(x)| \leq \frac{2A_1}{\sqrt{\lambda(2^{j+1})}} 2^j = \frac{2A_1}{\sqrt{\lambda(2^{j+1})}} \cdot |I_j| \leq \delta |I_j|, \quad x \in I_j,$$

and

$$|g_j'(x)| \leq 1, \quad x \in I_j.$$

By the Basic Lemma, there exist functions G_j such that

$$G_j \geq g_j, \quad G_j \equiv 0 \text{ outside } \frac{3}{2} I_j, \quad |\widetilde{G}_j'| \leq A_2\delta, \quad |G_j'| \leq A_2\delta,$$

$$\int\limits_{I_j} g_j(s)ds \geq A_3\delta \int\limits_{\mathbb{R}} G_j(s)ds, \quad G_j \geq 0, \quad G_j \in C^\infty(\mathbb{R}).$$

For the functions $H_j = G_j \cdot \lambda(2^{j+1})$, the following relations hold:

$$H_j \geq f_j \text{ on } I_j; \quad H_j \equiv 0 \text{ outside } \frac{3}{2}|I_j|,$$

$$|\widetilde{H}_j'| \leq A_2\delta\lambda(2^{j+1}), \quad |H_j'| \leq A_2\delta\lambda(2^{j+1}).$$

We construct similar functions H_j for $j < 0$. We take a function H_0 such that $H_0 \in C^\infty(\mathbb{R})$, H_0 is nonnegative, the support of H_0 is bounded, $H_0 \geq f_0$, and $|\widetilde{H}_0'| \leq \delta$. Set $\Omega_1 = \sum\limits_{j\in\mathbb{Z}} H_j$. Clearly, $\Omega_1 \geq \Omega$. Since the covering of the real line by intervals $\frac{3}{2}I_j$ has finite multiplicity and the function λ satisfies the doubling conditions (4) of Theorem 7, there exists an absolute constant A_4 such that

$$|\Omega_1'(x)| \le A_4\delta\lambda(x), \quad x \in \mathbb{R}.$$

It remains to estimate the function $\widetilde{\Omega_1}'(x)$. Take $x \in I_{j-1} \cup I_j \cup I_{j+1}$. Since the function λ satisfies condition (4), there exists a constant A_3 such that

$$|\widetilde{H}_j'(x)| \le A_3\delta\lambda(x) \ \text{ for } \ x \in I_{j-1} \cup I_j \cup I_{j+1}.$$

If $x \notin \frac{3}{2}I_j$, then

$$|\widetilde{H}_j'(x)| = \int\limits_{\mathbb{R}} \frac{H_j(t)}{(x-t)^2}dt \le 16 \int\limits_{\mathbb{R}} \frac{H_j(t)}{t^2} \tag{3.1}$$

since $|x-t| \ge \frac{|t|}{4}$.

Clearly $\int\limits_{\mathbb{R}} H_j \le A_4\delta \int\limits_{\mathbb{R}} f_j$. Since $\frac{1}{1+x^2} \asymp \frac{1}{1+y^2}$ for $x, y \in I_j$, the function Ω_1 belongs to $L^1(\mathbf{P})$. If $x \in I_{j_0}$, then

$$|\widetilde{\Omega_1}'(x)| \le |\widetilde{H}_0'(x)| + \sum_{|j|\ge N} |\widetilde{H_j}'(x)| \le \delta + \sum_{|j-j_0|\le 1, |j|\ge N} |\widetilde{H}_j'(x)| + \sum_{|j-j_0|>1, |j|\ge N} |\widetilde{H}_j'(x)|$$

$$\le \delta + 3A_3\delta\lambda(x) + 16 \sum_{|j-j_0|\ge 1, |j|\ge N} \int\limits_{\mathbb{R}} \frac{H_j(t)}{t^2}dt \le A_4\delta\lambda(x) + 16 \int\limits_{|t|\ge 2^{N-1}} \frac{\Omega_1(t)}{t^2}dt \le A_5\delta\lambda(x)$$

under a proper choice of N (since $\lambda \ge 1$). $\square$

Corollary 7.1. *Assume that functions Ω and λ satisfy the conditions of Theorem 7. If the inequality $\lambda(x) \le C(\lambda)|x|^\beta$, $x \in \mathbb{R}$, holds for some $\beta \in (0,1)$ and a constant $C(\lambda)$, then there exists a majorant Ω_1 that satisfies, in addition to conditions* (1)–(4) *of* (7), *the inequality*

$$|\widetilde{\Omega}''(x)| \le \varepsilon\lambda(x), \quad x \in \mathbb{R}. \tag{3.2}$$

Proof. Let us apply Theorem 7 to the functions Ω and λ and to a number ε_0 (the choice of $\varepsilon_0 = \varepsilon(\Omega, \lambda)$ is explained below) to construct a function Ω_1. Set $\Omega_2(x) = S\Omega_1(x) + C_1(\Omega_1)|x|^\beta + C_2(\Omega_1)$, where the constants $C_1(\Omega_1)$ and $C_2(\Omega_1)$ are taken to satisfy the inequality

$$\Omega_2(x) \ge \Omega_1(x), \quad x \in \mathbb{R}$$

(see the proof of Theorem 3). (Recall that S is the Steklov averaging operator.) Note that if $|x-y| < 1$ and $|x| > 2$, then the ratio $\lambda(x)/\lambda(y)$ is bounded. The same reasoning as in the proof of Theorem 3 shows that $S\Omega_1'(x) = \Omega_1'(s_1)$ and $\widetilde{(S\Omega_1)}'(x) = \widetilde{\Omega_1}'(s_2)$, where $s_1, s_2 \in [x-1, x+1]$. Note that

$$|\widetilde{(S\Omega_1)}''(x)| = \frac{1}{2}|\widetilde{\Omega_1}'(x+1) - \widetilde{\Omega_1}'(x-1)| \le \sup_{|s-x|\le 1} |\widetilde{\Omega_1}'(s)| \le \varepsilon_0\lambda(x+1) \le c(\lambda)\varepsilon_0\lambda(x).$$

Hence, the function $S\Omega_1$ satisfies inequalities (c) and (d) of Theorem 7 and condition (3.2) under a proper choive of ε_0. Decreasing ε_0 once more, we can satisfy inequalities (c), (d), and (3.2) for the function Ω_2 (and for arguments not less than a fixed number). It is easily seen that there exists a majorant of the function Ω_2 (with a converging Poisson integral) for which the above-mentioned inequalities hold on the whole real axis. The corollary is proved. $\square$

Recall that $h(x) = (f^{-1})'(x)$.

Theorem 8. *Assume that a differentiable function $f:\mathbb{R}\to\mathbb{R}$ is strictly increasing and satisfies the following conditions:*

$$(1)\qquad h(=h_f)\in \mathrm{Lip}_\beta(\mathbb{R}),\quad 0<\beta<1;\qquad (2)\qquad \int_{\mathbb{R}}\frac{dt}{1+f^2(t)}<+\infty;$$

$$(3)\qquad \lim_{|x|\to+\infty} f'(x)=0;\qquad (4)\qquad \sup_{x\in\mathbb{R}}\left|\frac{h(x)}{h(x/2)}\right|<+\infty;$$

$$5)\qquad \sup_{s\in[\min(0,x),\max(0,x)]} h(s)\le C_h h(x).$$

Then any smooth function $\omega=e^{-\Omega}$ such that

$$\text{(a)}\quad \Omega\in L^1(\mathbf{P});\qquad \text{(b)}\quad |\Omega'|\le Mh \quad \textit{for some}\quad M>0,$$

is admissible for the space K_{B_f}.

Proof. Since $h(x)=\frac{1}{f'(f^{-1}(x))}$, $\lim\limits_{|x|\to\infty} h(x)=+\infty$ (see condition (3) of Theorem 8). Set

$$\lambda(x)=C\sup_{s\in[\min(0,x),\max(0,x)]} h(s),$$

where the constant C is such that $\lambda\ge 1$. The function λ increases on the ray $[0,+\infty)$, decreases on the ray $(-\infty,0]$, and satisfies the doubling condition of Theorem 7.4 (since the function h satisfies this condition). Hence, for any $\varepsilon>0$ there exists a majorant Ω_1 such that $|\widetilde{\Omega_1}'|\le\varepsilon\lambda\le\varepsilon C_h h$ and $|\Omega_1'(x)|\le\varepsilon\lambda(x)\le\varepsilon C_h h(x)=O(|x|^\beta)$ (we take into account that the function h is Lipschitz continuous). We reduce ε so that $\varepsilon C_h<2\pi$. Now the statement of Theorem 8 follows from Theorem 4. $\square$

A differentiable, strictly increasing function f such that $f(x)=|x|^\alpha\log^\delta(|x|+2)$, $|x|\ge 1$ ($\frac12<\alpha<1$), satisfies the conditions of Theorem 7. A direct calculation shows that $h(x)\asymp|x|^{\frac{1}{\alpha}-1}\log^{-\frac{\delta}{\alpha}}(|x|+2)$.

Corollary 8.1. *A function $\omega=e^{-\Omega}$ such that*

$$\text{(a)}\qquad \int_{\mathbb{R}}\Omega\,d\mathbf{P}<+\infty;$$

$$\text{(b)}\qquad |\Omega'(x)|\le M|x|^{\frac{1}{\alpha}-1}\log^{-\frac{\delta}{\alpha}}(|x|+2)\quad \textit{for some } M>0 \textit{ and any } x\in\mathbb{R},$$

is admissible for the space K_{B_f}, where $f(x)=\operatorname{sgn}(x)|x|^\alpha(\log(|x|+2))^\delta$ ($\frac12<\alpha<1,\delta\in\mathbb{R}$).

§4. Admissibility conditions in the case of growth of imaginary parts of zeros of the Blaschke product

Consider a Blaschke product B with zeros $z_k=f(k)+ig(k)$, $k\in\mathbb{Z}$, where the function $f:\mathbb{R}\to\mathbb{R}$ is differentiable and strictly increasing, $\lim\limits_{x\to\pm\infty}f(x)=\pm\infty$, the function g is positive, and $\sum\limits_{k\in\mathbb{Z}}\frac{g(k)}{f^2(k)+g^2(k)}<+\infty$. By (1.1),

$$\frac{d}{dx}\arg B(x)=2\sum_{-\infty}^{+\infty}\frac{g(k)}{(x-f(k))^2+g^2(k)}=2\sigma(x). \tag{4.1}$$

Assume that

$$\text{(4a)}\quad \inf_{s\in\mathbb{R}}g(s)>0;\qquad \text{(4b)}\quad \sup_{|x-y|\le1}\frac{g(x)}{g(y)}<+\infty;\qquad \text{(4c)}\quad f\in\mathrm{Lip}_1(\mathbb{R}).$$

We claim that

$$\sigma(x)\asymp\int_{\mathbb{R}}\frac{g(t)}{(x-f(t))^2+g^2(t)}dt \tag{4.2}$$

in this case.

We need the following simple auxiliary statement.

Proposition 4. *Assume that numbers a, b, c, and d satisfy the inequalities $0 < M_0 < \frac{b}{d} < M_1$, $b > M_2 > 0$, $d > M_2$, and $|a - c| < M_3$. Then the ratio $\frac{a^2+b^2}{c^2+d^2}$ is separated from zero and infinity by constants that depend on the values M_0, M_1, M_2, and M_3 only.*

Proof. If $|a| > 1$ and $|c| > 1$, then $a \asymp c$ and $b \asymp d$. The desired statement follows from the elementary inequalities

$$\min\left(\frac{a^2}{c^2}, \frac{b^2}{d^2}\right) \le \frac{a^2+b^2}{c^2+d^2} \le \max\left(\frac{a^2}{c^2}, \frac{b^2}{d^2}\right).$$

If either $|a| \le 1$ or $|c| \le 1$, then $a, c \preceq 1$; hence, $a^2 + b^2 \asymp b^2$, $c^2 + d^2 \asymp d^2$, and $\frac{a^2+b^2}{c^2+d^2} \asymp \frac{b^2}{d^2}$. □

Set $F_x(t) = \frac{g(t)}{(x-f(t)^2)+g^2(t)}$, then $\sigma(x) = \sum\limits_{k\in\mathbb{Z}} F_x(k)$. On the other hand, if $|t_1 - t_2| \le 1$, then

$$\frac{F_x(t_1)}{F_x(t_2)} = \frac{g(t_1)}{g(t_2)} \cdot \frac{(x - f(t_2))^2 + g^2(t_2)}{(x - f(t_1))^2 + g^2(t_1)}.$$

The first fraction is separated from zero and infinity by condition (4b). Since the function f is Lipschitz continuous, the difference $|(x-f(t_1))-(x-f(t_2))|$ is bounded. By Proposition 4 (with $a = x-f(t_1)$, $c = x-f(t_2)$, $b = g(t_1)$, and $d = g(t_2)$), the second fraction is also separated from zero and infinity (by constants that do not depend on t_1, t_2, and x), i.e.,

$$0 < \inf_{x\in\mathbb{R},|t_1-t_2|\le 1} \frac{F_x(t_1)}{F_x(t_2)} \le \sup_{x\in\mathbb{R},|t_1-t_2|\le 1} \frac{F_x(t_1)}{F_x(t_2)} < +\infty.$$

Hence, $F_x(t) \asymp \int\limits_t^{t+1} F_x(s)ds$. Taking the sum of these estimates, we get the desired estimate (4.2):

$$\sigma(x) = \sum_{k\in\mathbb{Z}} F_x(k) \asymp \int\limits_{\mathbb{R}} F_x(t)dt.$$

Set

$$G(x) = g(f^{-1}(x)) \text{ and } H(x) = h(x)G(x) \text{ (recall that } h = (f^{-1})'). \tag{4.3}$$

Assume that $H \in \mathrm{Lip}_\gamma(\mathbb{R})$ for some $\gamma \in (0, 1)$ and that the function g satisfies conditions (4a) and (4b). Assume that $x > 0$ and $f(0) = 0$. Let us estimate the integral

$$\int\limits_0^{f^{-1}(x)} \frac{g(t)}{(x - f(t))^2 + g^2(t)}dt = \int\limits_0^x \frac{H(x - z)}{z^2 + G^2(x - z)}dz = H(x)\int\limits_0^x \frac{dz}{z^2 + G^2(x - z)} + \int\limits_0^x \frac{H(x - z) - H(x)}{z^2 + G^2(x - z)}dz.$$

We begin with the second integral:

$$\left|\int\limits_0^x \frac{H(x - z) - H(x)}{z^2 + G^2(x - z)}dz\right| \le \int\limits_0^x \frac{|H(x - z) - H(x)|}{z^2 + G^2(x - z)}dz \le C_H \int\limits_0^x \frac{z^\gamma}{z^2 + G^2(x - z)}dz \le C_1.$$

The last inequality is valid since the function G is separated from zero. Further,

$$\int\limits_0^x \frac{dz}{z^2 + G^2(x - z)} = \frac{1}{G(x)}\int\limits_0^x \frac{d(\frac{z}{G(x)})}{(\frac{z}{G(x)})^2 + (\frac{G(x-z)}{G(x)})^2} = \frac{1}{G(x)}\int\limits_0^{\frac{x}{G(x)}} \frac{dz}{z^2 + (\frac{G(x-G(x)z)}{G(x)})^2}.$$

Assume that $G(x) = o(x)$ and $\frac{G(x)}{G(y)} \to 1$ as $\frac{x}{y} \to 1$. Then

$$\int\limits_0^1 \frac{dz}{z^2 + (\frac{G(x-G(x)z)}{G(x)})^2} \preceq 1$$

since the ratio $\frac{x-G(x)z}{x}$ tends to 1 as $|x| \to +\infty$ uniformly in $z \in [0,1]$. On the other hand, we may take a so small that $\frac{G(x-G(x)z)}{G(x)} < 2$ for all $z \in [0, ax/G(x)]$. Hence,

$$\int_1^{\frac{x}{G(x)}} \frac{dz}{z^2 + (\frac{G(x-G(x)z)}{G(x)})^2} \geq \int_0^{a\frac{x}{G(x)}} \frac{dz}{z^2+4} \asymp 1$$

and

$$\int_1^{\frac{x}{G(x)}} \frac{dz}{z^2 + (\frac{G(x-G(x)z)}{G(x)})^2} \leq \int_1^{+\infty} \frac{dz}{z^2} = 1.$$

It follows that

$$\int_0^{\frac{x}{G(x)}} \frac{dz}{z^2 + (\frac{G(x-G(x)z)}{G(x)})^2} \asymp 1.$$

Thus,

$$\int_0^{f^{-1}(x)} \frac{g(t)}{(x-f(t))^2 + g^2(t)} dt \asymp \frac{H(x)}{G(x)} = h(x).$$

We apply similar reasoning to estimate the integral $\int\limits_{f^{-1}(x)}^{+\infty} \frac{g(t)}{(x-f(t))^2+g^2(t)} dt$. It is easily seen that the integral $\int\limits_{-\infty}^{0} \frac{g(t)}{(x-f(t))^2+g^2(t)} dt$ is bounded.

Thus (see (4.2)), if $H \in \mathrm{Lip}_\gamma(\mathbb{R})$ $(\gamma < 1)$, $G(x) = o(x)$, $\frac{G(x)}{G(y)} \to 1$ as $\frac{x}{y} \to 1$, and the function g satisfies conditions (4a) and (4b), then

$$\frac{d}{dx} \arg B(x) \asymp h(x). \tag{4.4}$$

We apply estimate (4.4) to prove the following statement.

Theorem 9. *Assume that a differentiable, strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ satisfies conditions* (1), (3), (4), *and* (5) *of Theorem* 8. *Assume that a function g satisfies condition* (4a), $\sum\limits_{k\in\mathbb{R}} \frac{g(k)}{f^2(k)+g^2(k)} < +\infty$ *(condition of convergence of the Blaschke product), and estimate* (4.4) *for the rate of growth of the argument of the Blaschke product is valid. Then any function $\omega = e^{-\Omega}$ such that*

(a) $\Omega \in L^1(\mathbf{P})$*;*

(b) $|\Omega'| \leq Mh$ *for some M,*

is admissible for the space K_B.

To prove Theorem 9, we need the following lemma.

Lemma. *Let $m : \mathbb{R} \to \mathbb{R}$ be a twice differentiable, strictly increasing function with the following properties:*

(1) $\lim\limits_{x\to\pm\infty} m'(x) = +\infty$*;*

(2) *there exists a constant C such that $|m''(x)| \leq Cm'(x)$ for any $x \in \mathbb{R}$.*

Then the function m is mainly increasing.

Proof. It is easily seen that there exists a sequence d_n such that $m(d_n) = 2\pi n$, $n \in \mathbb{Z}$. Set $I_n = [d_n, d_{n+1}]$ and $l_n = d_{n+1} - d_n$. By condition (1), $|I_n|_{n\to\pm\infty} \to 0$. Further,

$$\begin{aligned} \frac{1}{l_n} \int_{I_n} |m'(x) - m'(t)| dt &= \int_{I_n} |m''(\xi(t))| dt \leq C \int_{I_n} m'(\xi(t)) dt \\ \leq C \sup_{I_n}(m') l_n &= C \sup_{I_n}(m') \cdot (m^{-1})'(s) = C \frac{\sup_{I_n}(m')}{m'(t)}, \end{aligned} \tag{4.5}$$

where $x \in I_n$, $\xi(t) \in I_n$, $s \in (2\pi n, 2\pi n + 2\pi)$, and $t \in I_n$.

On the other hand, if x, $y \in I_n$, then

$$|m'(x) - m'(y)| = |m''(s)l_n| \le C \sup_{I_n}(m')l_n \Rightarrow |1 - m'(x)/\sup_{I_n} m'| \le l_n \to_{|n|\to+\infty} 0$$

uniformly in $x \in I_n$. Hence, if $|n|$ is large enough, then $\frac{\sup_{I_n} m'}{\inf_{I_n} m'} \le 2$. Now the statement of our lemma follows from (4.5). $\square$

Proof of Theorem 9. It suffices to show that the function $\arg B_{z_k} + 2\widetilde{\Omega}_1$ is mainly increasing for some majorant $\Omega_1 \ge \Omega$ with a finite Poisson integral. For any $\varepsilon > 0$ (we explain the choice of ε below) there exists a majorant Ω_1 such that $|\widetilde{\Omega_1}'| \le \varepsilon h$ and $|\widetilde{\Omega_1}''| \le \varepsilon h$. Indeed, we can apply the same reasoning as in the proof of Theorem 8 to construct a functon $\lambda(= C \sup_{s\in[\min(0,x),\max(0,x)]} h(s))$ that satisfies the conditions of Corollary 7.1 since $\lambda(x) = O(|x|^{\beta})$. Let us differentiate series (4.1):

$$\left|\frac{d^2}{dx^2}\arg B_{z_k}(x)\right| = \left|\sum_k \frac{4g(k)(x-f(k))}{((x-f(k)^2)+g^2(k))^2}\right| \le \sum_k \frac{2g(k)|x-f(k)|}{(x-f(k))^2+g^2(k)}\frac{2g(k)}{(x-f(k))^2+g^2(k)}\cdot\frac{1}{g(k)}$$
$$\le \frac{1}{\inf_{\mathbb{R}}(g)}\sum_k \frac{2g(k)}{(x-f(k))^2+g^2(k)} = \frac{1}{\inf_{\mathbb{R}}(g)}\frac{d}{dx}\arg B(x).$$

Thus, there exists a constant C_3 such that

$$\left|\frac{d^2}{dx^2}\arg B\right| \le C_3\frac{d}{dx}\arg B. \tag{4.6}$$

If ε is small enough, then $(\arg B + 2\widetilde{\Omega}_1)' \asymp h$. Hence, condition (1) of the above lemma is satisfied. We apply inequality (4.6) to fix ε such that

$$(\arg B + 2\widetilde{\Omega}_1)'' \preceq h \asymp (\arg B + 2\widetilde{\Omega}_1)'.$$

By our lemma, the function $\arg B + 2\widetilde{\Omega}_1$ is mainly increasing. $\square$

Consider the functions $f(x) = |x|^{\alpha}$, $\frac{1}{2} < \alpha < 1$, and $g(x) = |x|^{\gamma}$. It is easily seen that if $\gamma < 2\alpha - 1$, then this pair of functions satisfies the conditions of Theorem 9 (thus, the faster the function f grows, the faster the function g may grow). The following statement is a particular illustration of Theorem 9.

Corollary 9.1. *Let $z_k = \operatorname{sgn}(k)|x|^{\alpha} + i|x|^{\gamma}$, $k \ne 0$, where $\frac{1}{2} < \alpha < 1$ and $0 \le \gamma < 2\alpha - 1$. Then any function $\omega = e^{-\Omega}$ such that*

(a) $\int_{\mathbb{R}} \Omega d\,\mathbf{P} < +\infty$;

(b) $|\Omega'(x)| \le M|x|^{\frac{1}{\alpha}-1}$ *for some M and all $x \in \mathbb{R}$,*

is admissible for the space $K_{B_{z_k}}$.

If $g \equiv 1$, we again get Theorem 8.

Remark 5. If $0 < \inf_{\mathbb{R}} \frac{g_1}{g} \le \sup_{\mathbb{R}} \frac{g_1}{g} < +\infty$ for a function g_1, then Theorem 9 remains valid for the Blaschke product constructed by the zeros $z_k = f(k) + ig_1(k)$, $k \in \mathbb{Z}$.

Proof. It follows from Remark 2 that the function $(\arg B)'$ satisfies the required estimate, and inequality (4.6) remains valid for the new Blaschke product. $\square$

Remark 6. If an inner function B has the properties described in Theorem 9, then any majorant $\omega = e^{-\Omega}$ such that the logarithmic integral $\mathcal{L}(\omega)$ converges and Ω is Lipschitz continuous (i.e., such as in the Beurling–Malliavin theorem) is B-admissible.

Proof. Since the continuous function h is separated from zero, we can take M so large that

$$\sup_{s\in\mathbb{R}} |\Omega'(s)| < Mh.$$

Now Theorem 9 implies our statement. $\square$

Remark 7. If an inner function B has the properties described in Theorem 9, then any majorant $\omega = e^{-\Omega}$ that is even, positive, and decreasing on the positive half-axis is B-admissible.

Proof. For such a function Ω one can construct a Lipschitz continuous majorant Ω_1 that is summable with respect to the Poisson measure (see [5, p. 310), i.e., $|\Omega_1'| \preceq 1$. Hence, this majorant is admissoble. $\square$

§5. Admissibility conditions in the case of decrease of the imaginary parts of zeros of the Blaschke product

Similarly to Sec. 4, consider a Blaschke product with zeros $f(k)+ig(k)$, $k \in \mathbb{Z}$. The function f is differentiable and strictly increasing; the function g is positive and such that $\sum\limits_{k\in\mathbb{Z}} \frac{g(k)}{f^2(k)+g^2(k)} < +\infty$. In this section, we consider functions g such that $\inf_{s\in\mathbb{R}} g(s) = 0$. The difficulty of the case under consideration is that, in contrast to the cases of Secs. 3 and 4, the function $\arg B$ is not, in general, mainly increasing. Thus, we cannot use Theorem 1 (which was applied to prove all the admissibility conditions of this paper) to prove admissibility even for the majorant $\omega \equiv 1$ ($\Omega \equiv 0$). For such products, estimate (4.6) is not valid, and we cannot refer to the lemma from the previous section.

We need the following easy statement (similar to Proposition 4).

Proposition 5. *Assume that numbers a, b, c, and d satisfy the inequalities $0 < M_0 < \frac{b}{d} < M_1$, $0 < b < M_2$, $0 < d < M_2$, and $|a - c| < M_3 b$. Then the ratio $\frac{a^2+b^2}{c^2+d^2}$ is separated from zero and infinity by constants that depend on the values M_0, M_1, M_2, and M_3 only.*

Indeed, if $|a| > b$ and $|c| > b$, then $a \asymp c$ and $b \asymp d$. We apply the inequality

$$\min\left(\frac{a^2}{c^2}, \frac{b^2}{d^2}\right) \le \frac{a^2+b^2}{c^2+d^2} \le \max\left(\frac{a^2}{c^2}, \frac{b^2}{d^2}\right)$$

to prove the required statement. If either $|a| < b$ or $|c| < b$, then $a, c \preceq b$. Hence, $a^2 + b^2 \asymp b^2$ and $c^2 + d^2 \asymp d^2$; thus, $\frac{a^2+b^2}{c^2+d^2} \asymp \frac{b^2}{d^2}$. $\square$

Define a function σ by formula (4.1). Assume that

(5a) $\sup\limits_{|x-y|\le 1} \frac{g(x)}{g(y)} < +\infty$;

(5b) $f' \preceq g$.

Below we show that

$$\sigma(x) \asymp h(x) = \int\limits_{\mathbb{R}} \frac{g(t)}{(x - f(t))^2 + g^2(t)} dt \tag{5.1}$$

in this case. (In contrast to the previous sections, now the function h is, by definition, equal to the integral on the right in (5.1).) Set $F_x(t) = \frac{g(t)}{(x-f(t)^2)+g^2(t)}$. Then $\sigma(x) = \sum\limits_{k\in\mathbb{Z}} F_x(k)$. On the other hand, if $|t_1 - t_2| \le 1$, then

$$\frac{F_x(t_1)}{F_x(t_2)} = \frac{g(t_1)}{g(t_2)} \cdot \frac{(x - f(t_2))^2 + g^2(t_2)}{(x - f(t_1))^2 + g^2(t_1)}.$$

The first fraction is separated from zero and infinity by condition (5a). Since the function f is Lipschitz continuous, $|(x - f(t_1)) - (x - f(t_2))| \preceq g$ (see property (5b)). We apply Proposition 5 (with $a = x - f(t_1)$, $c = x - f(t_2)$, $b = g(t_1)$, and $d = g(t_2)$) to show that the second fraction is also separated from zero and infinity (by constants that do not depend on t_1, t_2, and x). Thus,

$$0 < \inf_{x\in\mathbb{R}, |t_1-t_2|\le 1} \frac{F_x(t_1)}{F_x(t_2)} \le \sup_{x\in\mathbb{R}, |t_1-t_2|\le 1} \frac{F_x(t_1)}{F_x(t_2)} < +\infty.$$

Hence, $F_x(t) \asymp \int\limits_t^{t+1} F_x(s)ds$. Summing the estimates, we get the required relation.

Theorem 10. *Let f be a differentiable, strictly increasing function and let g be a function such that*

$$\sum_{k\in\mathbb{Z}} \frac{g(k)}{f^2(k) + g^2(k)} < +\infty.$$

Let a function $h = (h_{f,g})$ satisfy conditions (1), (4), and (5) of Theorem 8 and let $\lim\limits_{|x|\to+\infty} h(x) = +\infty$. Assume that there exist constants $C_1, C_2 > 0$ such that

$$|\arg B(x) - \arg B(y)| \le C_1 \Rightarrow |(\arg B)'(x) - (\arg B)'(y)| \le C_2.$$

Then any smooth function $\omega = e^{-\Omega}$ *such that*

(a) $\Omega \in L^1(\mathbf{P})$;

(b) $|\Omega'| \le Mh$ *for some* $M > 0$,

is admissible for the space K_B.

Proof. Similarly to the proof of Theorem 9, for any $\varepsilon > 0$ we can find a function $\Omega_1 \ge \Omega$ such that $\Omega_1 \in L^1(\mathbf{P})$, $|\Omega_1'| \le \varepsilon h$, $|\widetilde{\Omega_1}'| \le \varepsilon h$, and $|\widetilde{\Omega_1}''| \le \varepsilon h$. We assume that the function Ω has the above properties. Set $a = \arg B$. We can take ε so small that

$$|\widetilde{\Omega}(x) - \widetilde{\Omega}(y)| \le \frac{1}{2}|a(x) - a(y)|, \quad x, y \in \mathbb{R}$$

(recall that $\arg B' \asymp h$). Then

$$|a(x) + 2\widetilde{\Omega}(x) - (a(y) + 2\widetilde{\Omega}(y))| \le \frac{C_1}{2} \Rightarrow |a(x) - a(y)| \le C_1. \tag{5.2}$$

It is easily seen that for such x, y (with $|a(x) + 2\widetilde{\Omega}(x) - (a(y) + 2\widetilde{\Omega}(y))| \le \frac{C_1}{2}$), the relation $|\widetilde{\Omega}'(x) - \widetilde{\Omega}'(y)| \le |x - y|\widetilde{\Omega}''(s) \preceq |x - y|h(s)$ holds (where s lies between x and y). The function h is separated from zero, the difference $|x - y|$ is bounded, and the ratio $\frac{h(s)}{h(t)}$ (where s and t lie between x and y) is bounded as well (this follows from properties (4) and (5) of the function h). Let us estimate the term $|x - y|h(s)$, where s lies between x and y, as follows:

$$|x - y|h(s) = |a^{-1}(a(x)) - a^{-1}(a(y))|h = (a^{-1})'(\zeta)|a(x) - a(y)|h(s) \preceq \frac{h(s)}{a'(t)}$$

for some t between x and y. Since $a' \asymp h$,

$$|\widetilde{\Omega}'(x) - \widetilde{\Omega}'(y)| \preceq 1.$$

Hence, there exists a constant C_3 such that

$$|a(x) + 2\widetilde{\Omega}(x) - (a(y) + 2\widetilde{\Omega}(y))| \le \frac{C_1}{2} \Rightarrow |a'(x) + 2\widetilde{\Omega}'(x) - (a'(y) + 2\widetilde{\Omega}'(y))| \le C_3.$$

If $a(d_n) + 2\widetilde{\Omega}(d_n) = 2\pi n$, $n \in \mathbb{Z}$, for a sequence $\{d_n\}$, then $\sup\limits_n\{\sup\limits_{s,t \in (d_n, d_{n+1})}\{a'(s) + 2\widetilde{\Omega}'(s) - (a'(t) + 2\widetilde{\Omega}'(t))\}\} < +\infty$. Hence, the function $\arg B + 2\widetilde{\Omega}$ is mainly increasing. $\square$

Note that the pair of functions $f(x) = |x|^\alpha$ and $g(x) = |x|^\beta$, where $\alpha \in (\frac{1}{2}, 1)$ and $\alpha - 1 < \beta < 0$, satisfies the conditions of Theorem 10.

Corollary 10.1. *Assume that* $z_k = \operatorname{sgn}(k)|x|^\alpha + i|x|^\beta$, $k \ne 0$, *where* $\frac{1}{2} < \alpha < 1$ *and* $\alpha - 1 < \beta \le 0$. *Then any function* $\omega = e^{-\Omega}$ *such that*

(a) $\int\limits_{\mathbb{R}} \Omega d\mathbf{P} < +\infty$;

(b) $|\Omega'(x)| \le M|x|^{\frac{1}{\alpha} - 1}$ *for some* M *and any* $x \in \mathbb{R}$,

is admissible for the space K_B.

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REFERENCES

1. A. D. Baranov, "Polynomials in the de Branges spaces of entire functions," *Ark. Mat.*, **44**, 16–38 (2006).
2. Yu. S. Belov, "Necessary admissibility conditions for some classes of model subspaces" (in preparation).

3. A. D. Baranov and V. P. Havin, “Admissible majorants for model subspaces and arguments of interior functions,” *Funkts. Anal. Prilozhen.*, **40**, 2–21 (2006).
4. A. D. Baranov, A. A. Borichev, and V. P. Havin, “Admissible majorants for meromorphic functions with fixed poles,” *Indiana University Math. J.* (to appear).
5. V. Havin and B. Jöricke, *The Uncertainty Principle in Harmonic Analysis*, Springer-Verlag (1994).
6. V. P. Havin and J. Mashreghi, “Admissible majorants for model subspaces of H^2. Part 1: slow winding of the generating inner function,” *Can. J. Math.*, **55**, 1231–1263 (2003).
7. V. P. Havin and J. Mashreghi, “Admissible majorants for model subspaces of H^2. Part 2: fast winding of the generating inner function,” *Can. J. Math.*, **55**, 1264–1301 (2003).
8. P. Koosis, *The Logarithmic integral.* I, Cambridge Stud. Adv. Math., **12**, Cambridge University Press, Cambridge (1988).
9. V. P. Havin, J. Mashreghi, and E. Nazarov, “The Beurling–Malliavin theorem on a multiplicator: the seventh proof,” *Algebra Analiz*, **17**, 3–68 (2005).